MAS 350/451/6352 Measure and Probability:
Solutions to Week 1 Problems, Chapter 1, Problems 1 to 3.

1. The case $n = 2$ is B(ii). Now suppose the result holds for some n. Then

$$\begin{aligned} A_1 \cup A_2 \cup \cdots \cup A_n \cup A_{n+1} &= (A_1 \cup A_2 \cup \cdots \cup A_n) \cup A_{n+1} \\ &= B_n \cup A_{n+1}. \end{aligned}$$

   Now $B_n = A_1 \cup A_2 \cup \cdots \cup A_n \in \mathbf{B}$ by the inductive hypothesis and $A_{n+1} \in \mathbf{B}$ by assumption. Hence $B_n \cup A_{n+1} \in \mathbf{B}$ by B(ii) and the result follows.

2. There are $\binom{n}{r}$ subsets of size r for $0 \leq r \leq n$ and so the total number of subsets is $\sum_{r=0}^{n} \binom{n}{r} = (1+1)^2 = 2^n$. Here we used the binomial theorem $(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$.

3. To show $\Sigma_1 \cap \Sigma_2$ is a σ-algebra we must verify S(i) to S(iii).

   S(i) Since $S \in \Sigma_1$ and $S \in \Sigma_2$, $S \in \Sigma_1 \cap \Sigma_2$.

   S(ii) Suppose (A_n) is a sequence of sets in $\Sigma_1 \cap \Sigma_2$. Then $A_n \in \Sigma_1$ for all $n \in \mathbb{N}$ and so $\bigcup_{n=1}^{\infty} A_n \in \Sigma_1$. But also $A_n \in \Sigma_2$ for all $n \in \mathbb{N}$ and so $\bigcup_{n=1}^{\infty} A_n \in \Sigma_2$. Hence $\bigcup_{n=1}^{\infty} A_n \in \Sigma_1 \cap \Sigma_2$.

   S(iii) If $A \in \Sigma_1 \cap \Sigma_2, A^c \in \Sigma_1$ and $A^c \in \Sigma_2$. Hence $A^c \in \Sigma_1 \cap \Sigma_2$.

   Note that the same argument can be used to show that if $\{\Sigma_n, n \in \mathbb{N})$ are all σ-algebras of subsets of S then so is $\bigcap_{n=1}^{\infty} \Sigma_n$.

   $\Sigma_1 \cup \Sigma_2$ is not in general a σ-algebra for if $A \in \Sigma_1$ and $B \in \Sigma_2$ there is no good reason why $A \cup B \in \Sigma_1 \cup \Sigma_2$. For example let $S = \{1, 2, 3\}, \Sigma_1 = \{\emptyset, \{1\}, \{2, 3\}, S\}, \Sigma_2 = \{\emptyset, \{2\}, \{1, 3\}, S\}, A = \{1\}, B = \{2\}$. Then $A \cup B = \{1, 2\}$ is neither in Σ_1 nor Σ_2.