MAS 350/451/6352 Measure and Probability:
Solutions to Week 3 Problems, Chapter 1, Problems 4, 7, 8.
Chapter 2 Problem 11

4. (a) $A \cup B = [A - (A \cap B)] \cup [B - (A \cap B)] \cup (A \cap B)$ is a disjoint union, hence using finite additivity and (1.3.2)

$$m(A \cup B) = m(A - A \cap B) + m(B - A \cap B) + m(A \cap B).$$

Then

$$\begin{aligned} m(A \cup B) + m(A \cap B) &= m(A - A \cap B) + m(B - A \cap B) + 2m(A \cap B) \\ &= [m(A - A \cap B) + m(A \cap B)] \\ &+ [m(B - A \cap B) + m(A \cap B)] \\ &= m(A) + m(B), \end{aligned}$$

where we use the fact that A is the disjoint union of $A - A \cap B$ and $A \cap B$, and the analogous result for B. Note that the possibility that $m(A \cap B) = \infty$ is allowed for within this proof.

(b) $m(A \cup B) \leq m(A \cup B) + m(A \cap B) = m(A) + m(B)$ follows immediately from (a) as $m(A \cap B) \geq 0$. The general case is proved by induction. We've just established $n = 2$. Now suppose the result holds for some n. Then

$$\begin{aligned} m\left(\bigcup_{i=1}^{n+1} A_i\right) &= m\left(\bigcup_{i=1}^{n} A_i \cup A_{n+1}\right) \\ &\leq m\left(\bigcup_{i=1}^{n} A_i\right) + m(A_{n+1}) \\ &\leq \sum_{i=1}^{n} m(A_i) + m(A_{n+1}) = \sum_{i=1}^{n+1} m(A_i). \end{aligned}$$

7. The easiest way to see that m is a measure is to first use 5(a) and (b) and induction to show that if $m_1, m_2 \ldots, m_n$ are measures and $c_1, c_2, \ldots, c_n$ are non-negative numbers then $c_1 m_1 + c_2 m_2 + \cdots + c_n m_n$ is a measure. Now apply this with $m_j = \delta_{x_j} (1 \leq j \leq n)$. To get a probability measure we need $\sum_{j=1}^{n} c_j = 1$ for then, as δ_{x_j} is a probability measure for all $1 \leq j \leq n$, we have

$$m(S) = \sum_{j=1}^{n} c_j \delta_{x_j}(S) = \sum_{j=1}^{n} c_j = 1.$$

8. By definition $(a,b) \in \mathcal{B}(\mathbb{R})$. We've shown in the notes that $\{a\}, \{b\} \in \mathcal{B}(\mathbb{R})$ and so by S(ii), $[a,b] = \{a\} \cup (a,b) \cup \{b\} \in \mathcal{B}(\mathbb{R})$.

11 (a) Since for all $n \in \mathbb{N}, \sup_{k\geq n} a_k = -\inf_{k\geq n}(-a_k)$, we have

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup_{k\geq n} a_k = \lim_{n\to\infty}\left(-\inf_{k\geq n}(-a_k)\right) = -\liminf_{n\to\infty}(-a_n).$$

(b) Since for all $n \in \mathbb{N}, \sup_{k\geq n}(a_k + b_k) \leq \sup_{k\geq n} a_k + \sup_{k\geq n} b_k$, the result is obtained similarly to (a) by taking limits on both sides.

(c) Argue as in (b) noting that the inequality is reversed for inf, or use (a) and (b) to argue that

$$\begin{aligned}\liminf_{n\to\infty}(a_n + b_n) &= -\limsup_{n\to\infty}(-a_n - b_n) \\ &\geq -\limsup_{n\to\infty}(-a_n) - \limsup_{n\to\infty}(-b_n) \\ &= \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n.\end{aligned}$$

(d) Use the fact that for all $n \in \mathbb{N}, \sup_{k\geq n}(a_k b_k) \leq \left(\sup_{k\geq n} a_k\right)\left(\sup_{k\geq n} b_k\right)$ and argue as in (b).

(e) Use the fact that for all $n \in \mathbb{N}, \inf_{k\geq n}(a_k b_k) \geq (\inf_{k\geq n} a_k)(\inf_{k\geq n} b_k)$ and argue as in (d).

(f) Since $0 \leq \liminf_{n\to\infty}|a_n| \leq \limsup_{n\to\infty}|a_n| = 0$, we must have $\liminf_{n\to\infty}|a_n| = 0$ and so $0 = \liminf_{n\to\infty}|a_n| = \limsup_{n\to\infty}|a_n|$ from which it follows that $\lim_{n\to\infty}|a_n| = 0$ and hence $\lim_{n\to\infty} a_n = 0$.