MAS 350/451/6352 Measure and Probability:
Solutions to Week 4 Problems, Chapter 2 Problems 10, 12, 20

10 (a) If $x \in A$ and $x \in B$, lhs $= 1$ and rhs$= 1 + 1 - 1 = 1$,
If $x \in A$ and $x \notin B$, lhs $= 1$ and rhs$= 1 + 0 - 0 = 1$,
If $x \notin A$ and $x \in B$, lhs $= 1$ and rhs$= 0 + 1 - 0 = 1$,
If $x \notin A$ and $x \notin B$, lhs $= 0$ and rhs$= 0 + 0 - 0 = 0$, and so we have equality of lhs and rhs in all possible cases.

(b) If $x \in A, x \notin A^c$ so lhs $= 1$ and rhs $= 1 - 0 = 1$,
if $x \notin A, x \in A^c$ so lhs $= 0$ and rhs $= 1 - 1 = 0$.

(c) Since $A = B \cup (A - B)$ and $B \cap (A - B) = \emptyset$, we can apply (a) to find that $\mathbf{1}_A = \mathbf{1}_B + \mathbf{1}_{A-B}$.

(d) The lhs and rhs are both non-zero only in the case where $x \in A$ and $x \in B$ when both lhs and rhs are 1.

For the last part, if $x \notin A$ then $x \notin A_n$ for all $n \in \mathbb{N}$ and so lhs = rhs = 0. If $x \in A$ then $x \in A_n$ for one and only one $n \in \mathbb{N}$ and so lhs = rhs = 1.

12 If $a \leq c, f^{-1}((a, \infty)) = S \in \Sigma$ and if $a > c, f^{-1}((a, \infty)) = \emptyset \in \Sigma$.

20 (a) Let A be the set of measure zero on which f_n fails to converge to f. Then $\lim_{n \to} f_n(x) = f(x)$ for all $x \in S - A$. But then by algebra of limits $\lim_{n \to} f_n(x)^2 = f(x)^2$ for all $x \in S - A$.

(b) A be the set of measure zero on which f_n fails to converge to f and B be the set of measure zero on which g_n fails to converge to g. Now $m(A \cup B) \leq m(A) + m(B) = 0$ and by algebra of limits $\lim_{n \to}(f_n(x) + g_n)(x) = f(x) + g(x)$ for all $x \in S - (A \cup B)$.

(c) This follows by writing $f_n g_n = \frac{1}{4}[(f_n + g_n)^2 - (f_n - g_n)^2]$ and using the results of (a) and (b).