MAS 350/451/6352 Measure and Probability: Solutions to Week 7 Problems, Chapter 2 Problems 15, 16, 18, 19, (21), Chapter 3 Problem 22.

15 $(g \circ f)^{-1}((a, \infty)) = f^{-1}(g^{-1}(a, \infty)$. Now g is Borel measurable and so $g^{-1}((a, \infty)) = A \in \mathcal{B}(\mathbb{R})$. Hence by Theorem 2.2.3, $f^{-1}(A) \in \Sigma$. So we conclude that $(g \circ f)^{-1}((a, \infty)) \in \Sigma$ and so $g \circ f$ is measurable.

If $X : \Omega \to \mathbb{R}$ is a random variable then it is a measurable function from $(\Omega, \mathcal{F})$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. If $g : \mathbb{R} \to \mathbb{R}$ is Borel measurable then $g(X) = g \circ X$ is again a random variable by what we have just shown. If g is not Borel measurable then we must be wary of interpreting $g(X)$ as a random variable, unless we can directly prove that it is measurable using some other technique.

16 Write $h = f \circ \tau_y$ where $\tau_y(x) = x + y$. The mapping τ_y is continuous and hence measurable and so h is measurable by Problem 15.

18 If f is differentiable then it is continuous and so measurable by Corollary 2.3.1. For each $x \in \mathbb{R}, f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$. Now $x \to f(x + h)$ is measurable by Problem 15, and $x \to \frac{f(x+h)-f(x)}{h}$ is measurable by Theorem 2.3.1 and Problem 14(b). Finally f' is measurable by Theorem 2.3.5.

19 There are 4 possibilities to consider for a given $c \in \mathbb{R}$: (i) If $f(x) = c$ then by monotonicity $f^{-1}((-\infty, c]) = (-\infty, x] \in \mathcal{B}(\mathbb{R})$. (ii) If $f(x) < c$ for all x then $f^{-1}((-\infty, c]) = \mathbb{R} \in \mathcal{B}(\mathbb{R})$. (iii) If f has a discontinuity so that c is not in its range, let $\alpha = \sup\{x \in \mathbb{R}; f(x) \leq c\}$ then $f^{-1}((-\infty, c]) = (-\infty, \alpha) \in \mathcal{B}(\mathbb{R})$. (iv) If $f(x) > c$ for all x then $f^{-1}((-\infty, c]) = \emptyset \in \mathcal{B}(\mathbb{R})$.

21 (a) Its sufficient to consider the case where $x = a$. Then for any $\epsilon > 0$ and arbitrary $\delta, f(a - \delta) = 0 < f(a) + \epsilon = 1 + \epsilon$ and $f(a + \delta) = f(a) < f(a) + \epsilon = 1 + \epsilon$.

(b) Its sufficient to consider the case $x = n$ for some integer n. Again for any $\epsilon > 0$ and arbitrary $\delta, f(n - \delta) = n - 1 < f(n) + \epsilon = n + \epsilon$ and $f(n + \delta) = n < f(n) + \epsilon = n + \epsilon$.

(c) Let $U = f^{-1}((-\infty, a))$. We will show that U is open. Then it is a Borel set and f is measurable. Fix $x \in U$ and let $\epsilon = a - f(x)$. Then there exists $\delta > 0$ so that $|x - y| < \delta \Rightarrow f(y) < f(x) + \epsilon = a$ and so $y \in U$. We have shown that for each $x \in U$ there exists an open interval (of radius δ) so that if y is in this interval then $y \in U$. Hence U is open.

22 $f = \mathbf{1}_{[-2,-1)} + 2\mathbf{1}_{[0,1)} + \mathbf{1}_{[1,2)}$.
$\int_{\mathbb{R}} f(x)dx = 1 + 2 + 1 = 4.$