**MAS 350/451/6352 Measure and Probability: Solutions to Week 8 Problems, Chapter 3 Problems 23, 25, 26, 27, 28.**

23 If $f = \sum_{i=1}^{n} c_i \mathbf{1}_{A_i}$, $f\mathbf{1}_A = \sum_{i=1}^{n} c_i \mathbf{1}_{A_i}\mathbf{1}_A = \sum_{i=1}^{n} c_i \mathbf{1}_{A_i \cap A}$, by Problem 10(d).

If $f \geq 0, c_i \geq 0 (1 \leq i \leq n)$ and so $f\mathbf{1}_A \geq 0$. $I_A f = \sum_{i=1}^{n} c_i \mathbf{1} m(A_i \cap A)$ and if $m(A) < \infty$ then $I_A f < \infty$, since $m(A_i \cap A) < m(A) < \infty$ for all $1 \leq i \leq n$.

25 Imitating the proof of lemma 3.3.1., let $A = \{x \in S; |f(x)| \geq c\}$. Then

$$\int_S f^2 dm \geq \int_A f^2 dm \geq c^2 m(A),$$

and so $m(A) \leq \frac{1}{c^2} \int_S f^2 dm$.

The generalisation to $p \geq 1$ is

$$m(\{x \in S; |f(x)| \geq c\}) \leq \frac{1}{c^p} \int_S |f|^p dm,$$

and it is proved similarly. Note that when $p$ is odd, we need to replace $f$ by $|f|$ inside the integral to ensure non-negativity.

26 This follows immediately from the result of Problem 24, when you replace $f$ by $X - \mu$

27 Define $A = \{x \in S; f(x) \neq 0\} = \{x \in S; |f(x)|^p \neq 0\}$ and for each $n \in \mathbb{N}$ define $A_n = \{x \in S; |f(x)|^p \geq 1/n\}$. Then as in the proof of Corollary 3.3.1, $m(A) \leq \sum_{n=1}^{\infty} m(A_n)$ and by the last part of Problem 24, for each $n \in \mathbb{N}$,

$$m(A_n) \leq n \int_S |f|^p dm = 0.$$

28 $f_+ = \mathbf{1}_{[-1,0)} + 3\mathbf{1}_{[1,2)}, f_- = \mathbf{1}_{[-2,-1)} + 2\mathbf{1}_{[0,1)}$. $\int_{\mathbb{R}} f(x)dx = \int_{\mathbb{R}} f_+(x)dx - \int_{\mathbb{R}} f_-(x)dx = (1 + 3) - (1 + 2) = 1$.