MAS331: Metric Spaces
Solutions to Problems on Chapter 1 (but excluding those on Assignment 1)

1.

$$d_1((3,1,4),(2,7,1)) = |3-2| + |1-7| + |4-1| = 1+6+3 = 10.$$

$$d_2((3,1,4),(2,7,1)) = \sqrt{(3-2)^2 + (1-7)^2 + (4-1)^2} = \sqrt{1+36+9} = \sqrt{46}.$$

$$d_\infty((3,1,4),(2,7,1)) = \max(|3-2|, |1-7|, |4-1|) = 6.$$

2. In $\mathbb{R}^4$, $d_1((4,4,4,6),(0,0,0,0)) = 4+4+4+6 = 18$ and $d_1((3,5,5,5),(0,0,0,0)) = 3+5+5+5 = 18 = d_1((4,4,4,6),(0,0,0,0))$.
   $d_2((4,4,4,6),(0,0,0,0)) = \sqrt{4^2+4^2+4^2+6^2} = \sqrt{84}$ and $d_2((3,5,5,5),(0,0,0,0)) = \sqrt{3^2+5^2+5^2+5^2} = \sqrt{84} = d_2((4,4,4,6),(0,0,0,0))$.
   $d_\infty((4,4,4,6),(0,0,0,0)) = 6$ and $d_\infty((3,5,5,5),(0,0,0,0)) = 5 \neq d_\infty((4,4,4,6),(0,0,0,0))$.

3. Let $a = (a_1,\ldots,a_n), b = (b_1,\ldots,b_n) \in \mathbb{R}^n$.

   (a) For each i,

$$\begin{aligned} d_2(a,b) &= \sqrt{\sum_{1\le i\le n}(a_i-b_i)^2} \\ &\ge \sqrt{(a_i-b_i)^2} \\ &= |a_i-b_i|. \end{aligned}$$

   Therefore $d_2(a,b) \ge \max_{1\le i\le n}|a_i-b_i| = d_\infty(a,b)$.

$$\begin{aligned} d_1(a,b)^2 &= \left(\sum_{1\le i\le n}|a_i-b_i|\right)^2 \\ &= \sum_{1\le i\le n}|a_i-b_i|^2 + 2\sum_{1\le i<j\le n}|a_i-b_i||a_j-b_j| \\ &\ge \sum_{1\le i\le n}|a_i-b_i|^2 \\ &= d_2(a,b)^2. \end{aligned}$$

   Thus $d_1(a,b)^2 \ge d_2(a,b)^2$ and, taking square roots, $d_1(a,b) \ge d_2(a,b)$.

   (b) See Solutions to Assignment 1.

   (c) See Solutions to Assignment 1.

(d) From the proof of the first inequality in (a), $d_\infty(a,b) = d_2(a,b)$ if and only if $n-1$ of the numbers $(a_i - b_i)$ are 0. For example it happens for $(1,2,3)$ and $(\alpha,2,3)$ or $(1,\alpha,3)$ or $(1,2,\alpha)$ for any $\alpha \in \mathbb{R}$.

From the proof of the second inequality in (a), $d_2(a,b) = d_1(a,b)$ if and only if $\sum_{1\leq i<j\leq n} |a_i - b_i||a_j - b_j| = 0$. This happens if and only if $n-1$ of the numbers $(a_i - b_i)$ are 0, the same condition as for $d_\infty(a,b) = d_2(a,b)$.

(e) See Solutions to Assignment 1.

4. We need to check that d_1 satisfies the three axioms for a metric space.

We begin with M1. Let $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n)$ be two vectors in $\mathbb{R}^n$. Obviously $d_1(a,b) = 0$ if $a = b$. On the other hand, if $d_1(a,b) = 0$, this means that $|a_1 - b_1| + \cdots + |a_n - b_n| = 0$. Since any modulus is non-negative, we conclude that both $|a_i - b_i| = 0$ for all i, and thus $a = b$.

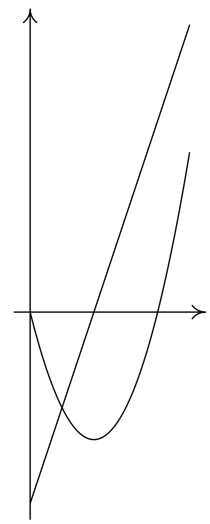
For M2, simply notice that

$$\begin{aligned} d_1(a,b) &= |a_1 - b_1| + \cdots + |a_n - b_n| \\ &= |b_1 - a_1| + \cdots + |b_n - a_n| \\ &= d_1(b,a). \end{aligned}$$

Finally we prove M3. Let $c = (c_1, \dots, c_n)$ be a third element of $\mathbb{R}^n$. Then

$$\begin{aligned} d_1(a,c) &= \sum |a_i - c_i| \\ &= \sum |(a_i - b_i) + (b_i - c_i)| \\ &\leqslant \sum |a_i - b_i| + |b_i - c_i| \\ &= \sum |a_i - b_i| + \sum |b_i - c_i| \\ &= d_1(a,b) + d_1(b,c). \end{aligned}$$

6. Here's a picture of the graphs between 0 and 5:



To compute $d_\infty(f_1, f_2)$, we need to find the maximum value for $|f(x) - g(x)|$ with $x \in I$. The only possible x values to consider are stationary points of $f - g$ and the two endpoints of the interval I.

At $x = 0$, $|f(x) - g(x)| = 6$; at $x = 5$, $|f(x) - g(x)| = 4$. Next, let's work out the stationary points of $f - g$. The difference $f(x) - g(x)$ is $x^2 - 7x + 6$, so its derivative is $2x - 7$; thus there is only one stationary point, at $x = \frac{7}{2}$, and the graphs are a distance of $\frac{25}{4}$ apart there. Since $\frac{25}{4} > 6$, this is the required answer. So

$$d_\infty(x^2 - 4x, 3x - 6) = \frac{25}{4}.$$

To compute $d_1(f_1, f_2)$, we calculate the area between the two graphs. Now the graphs meet when $x^2 - 4x = 3x - 6$, which occurs at $x = 1$ (and again outside the interval $[0, 5]$). We have $x^2 - 4x \geq 3x - 6$ for $0 \leqslant x \leqslant 1$, and $x^2 - 4x \leqslant 3x - 6$ for $1 \leqslant x \leqslant 5$. So

$$\begin{aligned}
d_1(x^2 - 4x, 3x - 6) &= \int_0^5 |(x^2 - 4x) - (3x - 6)|\, dx \\
&= \int_0^1 (x^2 - 4x) - (3x - 6)\, dx + \int_1^5 (3x - 6) - (x^2 - 4x)\, dx \\
&= \left[\frac{x^3}{3} - \frac{7x^2}{2} + 6x\right]_0^1 + \left[\frac{7x^2}{2} - \frac{x^3}{3} - 6x\right]_1^5 \\
&= \left(\frac{17}{6} - 0\right) + \left(\frac{95}{6} - \left(-\frac{17}{6}\right)\right) \\
&= \frac{129}{6} = \frac{43}{2}.
\end{aligned}$$

8.

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise} \end{cases}$$

It is clear from this definition that d satisfies M1 and M2.

It remains to verify the triangle inequality M3. Let $x, y, z \in X$. We want to show that $d(x,y) \leqslant d(x,z) + d(z,y)$. If $d(x,y) = 0$ then the inequality obviously holds. Therefore we assume that $d(x,y) = 1$ so that $x \neq y$. It follows that z must be different from one of x or y so that $d(x,z) = 1$ or $d(z,y) = 1$ or both. Hence

$$d(x,z) + d(z,y) \geqslant 1 = d(x,y).$$

9. Let $f, g \in C[a,b]$. Then, since $|f(x) - g(x)| = |g(x) - f(x)|$ for all $x \in [a,b]$, we have

$$\begin{aligned} d_1(f,g) &= \int_a^b |f(x) - g(x)| dx \\ &= \int_a^b |g(x) - f(x)| dx \\ &= d_1(g,f), \end{aligned}$$

so that d_1 satisfies axiom M2.

Now let $h \in C[a,b]$. Since

$$|f(x) - h(x)| = |(f(x) - g(x)) + (g(x) - h(x))| \leqslant |f(x) - g(x)| + |g(x) - h(x)|$$

for all $x \in [a,b]$, we have

$$\begin{aligned} d_1(f,h) &= \int_a^b |f(x) - h(x)| dx \\ &\leqslant \int_a^b |f(x) - g(x)| + |g(x) - h(x)| dx \\ &= \int_a^b |f(x) - g(x)| dx + \int_a^b |g(x) - h(x)| dx \\ &= d_1(f,g) + d_1(g,h), \end{aligned}$$

which proves M3.

11. First do the case $d(x,y) = \sqrt{|x-y|}$. Take $x, y, z \in \mathbb{R}$. For M1, first $d(x,x) = \sqrt{|x-x|} = 0$, and then if $\sqrt{|x-y|} = d(x,y) = 0$ we find $|x-y| = (\sqrt{|x-y|})^2 = 0$, so that $x = y$.

For M2, we simply note that $d(y,x) = \sqrt{|y-x|} = \sqrt{|x-y|} = d(x,y)$.

For M3, we want to know if

$$d(x,y) = \sqrt{|x-y|} \leqslant d(x,z) + d(z,y) = \sqrt{|x-z|} + \sqrt{|z-y|}.$$

If we square both sides, we end up asking if the inequality

$$|x-y| \leqslant |x-z| + |z-y| + 2\sqrt{|x-z||z-x|}$$

holds. The answer clearly is yes since we already have $|x-y| \leqslant |x-z| + |z-y|$.

In the second case, if we take $x = 1$, $y = -1$ and $z = 0$ we find that $d(x,y) = (1-(-1))^2 = 4$, but $d(x,z)+d(z,y) = 1^2+(-1)^2 = 2$. Thus M3 is not satisfied and this d is not a metric. (Here M1 and M2 are satisfied.)

12. For M1, we first check that $d(x,y) \geq 0$ for all $x, y \in X$: as d_1 and d_2 satisfy M1,

$$d(x,y) = d_1(x,y) + d_2(x,y) \geq 0.$$

Also, as d_1 and d_2 satisfy M1, $d(x,x) = d_1(x,x) + d_2(x,x) = 0 + 0 = 0$. And if $d(x,y) = 0$ then, as $d_1(x,y) \geq 0$ and $d_2(x,y) \geq 0$, we must have $d_1(x,y) = d_2(x,y) = 0$ and so $x = y$.

For M2, d_1 and d_2 satisfy M2 so

$$d(x,y) = d_1(x,y) + d_2(x,y) = d_1(y,x) + d_2(y,x) = d(y,x).$$

For M3, d_1 and d_2 satisfy M3 so, for $x, y, z \in X$,

$$\begin{aligned} d(x,z) &= d_1(x,z) + d_2(x,z) \\ &\leq (d_1(x,y) + d_1(y,z)) + (d_2(x,y) + d_2(y,z)) \\ &= (d_1(x,y) + d_2(x,y)) + (d_1(y,z) + d_2(y,z)) \\ &= d(x,y) + d(y,z). \end{aligned}$$

Thus d satisfies M1,2,3 and d is a metric on X.

13. $d_1((0,0),(1,0)) = 1 = d_2((0,0),(1,0))$ so $d((0,0),(1,0)) = 0$. As $(0,0) \neq (1,0)$, d fails axiom M1 and is therefore not a metric.

14. (a) Let $n = 2$. Then one of $|x_1 - y_1|, |x_2 - y_2|$ is $\max(|x_i - y_i|)$ and the other is $\min\{|x_i - y_i|\}$. This includes the case where $|x_1 - y_1| = |x_2 - y_2|$ and $\max(|x_i - y_i|) = \min(|x_i - y_i|)$. Hence

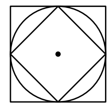
$$\max(|x_i - y_i|) + \min(|x_i - y_i|) = |x_1 - y_1| + |x_2 - y_2|,$$

that is, $e(x,y) = d_1(x,y)$.

(b) Let $x = (0,0,0), y = (1,2,3)$ and $z = (2,2,4)$.
Then $e(x,y)) = 3+1 = 4$, $e(y,z) = 1+0 = 1$. But $e(x,z) = 4+2 = 6 > e(x,y) + e(y,z)$ so axiom M3 fails and e is not a metric.

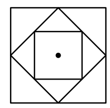
5. Let $r > 0$ and let $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in \mathbb{R}^n$. Suppose that $b \in B_1(a,r)$. Then $d_1(b,a) < r$. But $d_1(a,b) \geq d_2(a,b)$ by 3(a) so $d_2(a,b) \leq d_1(b,a) < r$ and hence $b \in B_2(a,r)$. Thus $B_1(a,r) \subseteq B_2(a,r)$.

Now suppose that $b \in B_2(a,r)$. Then $d_2(b,a) < r$. But $d_2(a,b) \geq d_\infty(a,b)$ by 3(a) so $d_\infty(a,b) \leq d_2(b,a) < r$ and hence $b \in B_\infty(a,r)$. Thus $B_2(a,r) \subseteq B_\infty(a,r)$.



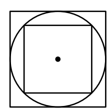
$$B_1(a,r) \subseteq B_2(a,r) \subseteq B_\infty(a,r)$$

Next, suppose that $b \in B_\infty(a, \frac{r}{n})$. Then $d_\infty(b,a) < \frac{r}{n}$. Using 3(b), $d_1(a,b) \leq n d_\infty(a,b) < r$ so $b \in B_1(a,r)$. Thus $B_\infty(a, \frac{r}{n}) \subseteq B_1(a,r)$.



$$B_\infty(a, \tfrac{r}{2}) \subseteq B_1(a,r) \subseteq B_\infty(a,r)$$

Finally, suppose that $b \in B_\infty(a, \frac{r}{\sqrt{n}})$. Then $d_\infty(b,a) < \frac{r}{\sqrt{n}}$. Using 3(c), $d_2(a,b) \leq \sqrt{n} d_\infty(a,b) < r$ so $b \in B_2(a,r)$. Thus $B_2(a, \frac{r}{\sqrt{n}}) \subseteq B_2(a,r)$.



$$B_\infty(a, \tfrac{r}{\sqrt{2}}) \subseteq B_2(a,r) \subseteq B_\infty(a,r)$$

$$B_\infty(a, \frac{r}{n}) \subseteq B_1(a,r) \text{ and } B_\infty(a, \frac{r}{\sqrt{n}}) \subseteq B_2(a,r).$$

Here, as in the notes, the subscripts indicate which metric we are using.

When $n > 1$, all the inclusions $\subseteq$ are strict inclusions.

7.

$$B[f,1] = \{g \in C[0,1] \mid d_\infty(f,g) \leqslant 1\},$$

and it follows from the definitions that

$$d_\infty(f,g) = \sup\{|g(x) - f(x)| \,:\, x \in [0,1]\} = \sup\{|g(x)| \,:\, x \in [0,1]\}$$

so that $d_\infty(f,g) \leqslant 1$ if and only $|g(x)| \leqslant 1$ for all $x \in [0,1]$. This means that the closed ball is the set of functions whose graph lies between the lines $y = 1$ and $y = -1$.

10. (a) $(4,2)$, $(2,1)$ and the origin are collinear so $d((4,2),(2,1)) = d_2((4,2),(2,1)) = \sqrt{5}$. $(4,2)$, $(-2,3)$ and the origin are not collinear so we have to go via the origin and

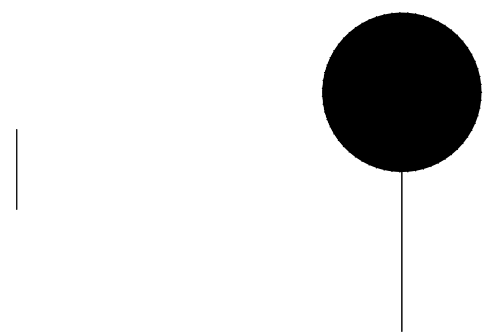
$$d((4,2),(-2,3)) = d_2((4,2),(0,0)) + d_2((-2,3),(0,0)) = 2\sqrt{5} + \sqrt{13}.$$

(b) $(0,-1)$ is at distance 1 from $(0,0)$ so the only points within $\frac{1}{2}$ of $(0,-1)$ are on the line through $(-1,0)$ and $(0,0)$ (the y-axis) so

$$B((0,-1),\frac{1}{2}) = \{(0,y) : -0.5 > y > -1.5\}.$$

There are two types of points in $B((0,-1),2)$. There are those that are collinear with $(-1,0)$ and $(0,0)$ at distance < 2 from $(-1,0)$. That gives $\{(0,y) : 1 > y > -3\}$. There are those that we reach via the $(0,0)$ which must be < 1 from $(0,0)$. That gives $\{(x,y) : x^2 + y^2 < 1\}$. So

$$B((0,-1),2) = \{(x,y) : x^2 + y^2 < 1 \text{ or } (x = 0 \text{ and } -3 < y < 1)\}.$$



$B((0,-1),\frac{1}{2})$ $B((0,-1),2)$

In the right-hand diagram $(-1,0)$ is included but all other points on the unit circle are excluded. In the left-hand diagram, both endpoints, $(0,-0.5)$ and $(0,-1.5)$, are excluded.