

MAS331: Metric Spaces 2016-17

Solutions to Week 1 Problems on Chapter 1: (1), (2), (3a,d), (4)

1.

$$d_1((3, 1, 4), (2, 7, 1)) = |3 - 2| + |1 - 7| + |4 - 1| = 1 + 6 + 3 = 10.$$

$$d_2((3, 1, 4), (2, 7, 1)) = \sqrt{(3 - 2)^2 + (1 - 7)^2 + (4 - 1)^2} = \sqrt{1 + 36 + 9} = \sqrt{46}.$$

$$d_\infty((3, 1, 4), (2, 7, 1)) = \max(|3 - 2|, |1 - 7|, |4 - 1|) = 6.$$

2. In \mathbb{R}^4 , $d_1((4, 4, 4, 6), (0, 0, 0, 0)) = 4+4+4+6 = 18$ and $d_1((3, 5, 5, 5), (0, 0, 0, 0)) = 3 + 5 + 5 + 5 = 18 = d_1((4, 4, 4, 6), (0, 0, 0, 0))$.

$$d_2((4, 4, 4, 6), (0, 0, 0, 0)) = \sqrt{4^2 + 4^2 + 4^2 + 6^2} = \sqrt{84} \text{ and } d_2((3, 5, 5, 5), (0, 0, 0, 0)) = \sqrt{3^2 + 5^2 + 5^2 + 5^2} = \sqrt{84} = d_2((4, 4, 4, 6), (0, 0, 0, 0)).$$

$$d_\infty((4, 4, 4, 6), (0, 0, 0, 0)) = 6 \text{ and } d_\infty((3, 5, 5, 5), (0, 0, 0, 0)) = 5 \neq d_\infty((4, 4, 4, 6), (0, 0, 0, 0)).$$

3. Let $a = (a_1, \dots, a_n), b = (b_1, \dots, b_n) \in \mathbb{R}^n$.

(a) For each i ,

$$\begin{aligned} d_2(a, b) &= \sqrt{\sum_{1 \leq i \leq n} (a_i - b_i)^2} \\ &\geq \sqrt{(a_i - b_i)^2} \\ &= |a_i - b_i|. \end{aligned}$$

Therefore $d_2(a, b) \geq \max_{1 \leq i \leq n} |a_i - b_i| = d_\infty(a, b)$.

$$\begin{aligned} d_1(a, b)^2 &= \left(\sum_{1 \leq i \leq n} |a_i - b_i| \right)^2 \\ &= \sum_{1 \leq i \leq n} |a_i - b_i|^2 + 2 \sum_{1 \leq i < j \leq n} |a_i - b_i| |a_j - b_j| \\ &\geq \sum_{1 \leq i \leq n} |a_i - b_i|^2 \\ &= d_2(a, b)^2. \end{aligned}$$

Thus $d_1(a, b)^2 \geq d_2(a, b)^2$ and, taking square roots, $d_1(a, b) \geq d_2(a, b)$.

(d) From the proof of the first inequality in (a), $d_\infty(a, b) = d_2(a, b)$ if and only if $n - 1$ of the numbers $(a_i - b_i)$ are 0. For example it happens for $(1, 2, 3)$ and $(\alpha, 2, 3)$ or $(1, \alpha, 3)$ or $(1, 2, \alpha)$ for any $\alpha \in \mathbb{R}$.

From the proof of the second inequality in (a), $d_2(a, b) = d_1(a, b)$ if and only if $\sum_{1 \leq i < j \leq n} |a_i - b_i| |a_j - b_j| = 0$. This happens if and only if $n - 1$ of the numbers $(a_i - b_i)$ are 0, the same condition as for $d_\infty(a, b) = d_2(a, b)$.

4. We need to check that d_1 satisfies the three axioms for a metric space.

We begin with M1. Let $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n)$ be two vectors in \mathbb{R}^n . Obviously $d_1(a, b) = 0$ if $a = b$. On the other hand, if $d_1(a, b) = 0$, this means that $|a_1 - b_1| + \dots + |a_n - b_n| = 0$. Since any modulus is non-negative, we conclude that both $|a_i - b_i| = 0$ for all i , and thus $a = b$.

For M2, simply notice that

$$\begin{aligned}d_1(a, b) &= |a_1 - b_1| + \dots + |a_n - b_n| \\ &= |b_1 - a_1| + \dots + |b_n - a_n| \\ &= d_1(b, a).\end{aligned}$$

Finally we prove M3. Let $c = (c_1, \dots, c_n)$ be a third element of \mathbb{R}^n . Then

$$\begin{aligned}d_1(a, c) &= \sum |a_i - c_i| \\ &= \sum |(a_i - b_i) + (b_i - c_i)| \\ &\leq \sum |a_i - b_i| + |b_i - c_i| \\ &= \sum |a_i - b_i| + \sum |b_i - c_i| \\ &= d_1(a, b) + d_1(b, c).\end{aligned}$$