MAS331/6352: Metric Spaces

Solutions to Week 1 Problems on Chapter 1: (1), (2), (3a,d), (4)

1.

$$d_1((3,1,4),(2,7,1)) = |3-2| + |1-7| + |4-1| = 1+6+3 = 10.$$

$$d_2((3,1,4),(2,7,1)) = \sqrt{(3-2)^2 + (1-7)^2 + (4-1)^2} = \sqrt{1+36+9} = \sqrt{46}.$$

$$d_\infty((3,1,4),(2,7,1)) = \max(|3-2|, |1-7|, |4-1|) = 6.$$

2. In $\mathbb{R}^4$, $d_1((4,4,4,6),(0,0,0,0)) = 4+4+4+6 = 18$ and $d_1((3,5,5,5),(0,0,0,0)) = 3+5+5+5 = 18 = d_1((4,4,4,6),(0,0,0,0))$.

   $d_2((4,4,4,6),(0,0,0,0)) = \sqrt{4^2+4^2+4^2+6^2} = \sqrt{84}$ and $d_2((3,5,5,5),(0,0,0,0)) = \sqrt{3^2+5^2+5^2+5^2} = \sqrt{84} = d_2((4,4,4,6),(0,0,0,0))$.

   $d_\infty((4,4,4,6),(0,0,0,0)) = 6$ and $d_\infty((3,5,5,5),(0,0,0,0)) = 5 \neq d_\infty((4,4,4,6),(0,0,0,0))$.

3. (a) Let $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in \mathbb{R}^n$.
   For each i,

$$\begin{aligned} d_2(a,b) &= \sqrt{\sum_{1\le i\le n} (a_i - b_i)^2} \\ &\ge \sqrt{(a_i-b_i)^2} \\ &= |a_i - b_i|. \end{aligned}$$

   Therefore $d_2(a,b) \ge \max_{1\le i\le n} |a_i - b_i| = d_\infty(a,b)$.

$$\begin{aligned} d_1(a,b)^2 &= \left(\sum_{1\le i\le n} |a_i - b_i|\right)^2 \\ &= \sum_{1\le i\le n} |a_i-b_i|^2 + 2\sum_{1\le i<j\le n} |a_i-b_i||a_j-b_j| \\ &\ge \sum_{1\le i\le n} |a_i-b_i|^2 \\ &= d_2(a,b)^2. \end{aligned}$$

   Thus $d_1(a,b)^2 \ge d_2(a,b)^2$ and, taking square roots, $d_1(a,b) \ge d_2(a,b)$.

   (d) From the proof of the first inequality in (a), $d_\infty(a,b) = d_2(a,b)$ if and only if $n-1$ of the numbers $(a_i - b_i)$ are 0. For example it happens for $(1,2,3)$ and $(\alpha,2,3)$ or $(1,\alpha,3)$ or $(1,2,\alpha)$ for any $\alpha \in \mathbb{R}$.

   From the proof of the second inequality in (a), $d_2(a,b) = d_1(a,b)$ if and only if $\sum_{1\le i<j\le n} |a_i-b_i||a_j-b_j| = 0$. This happens if and only if $n-1$ of the numbers $(a_i - b_i)$ are 0, the same condition as for $d_\infty(a,b) = d_2(a,b)$.

4. We need to check that d_1 satisfies the three axioms for a metric space.

   We begin with M1. Let $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n)$ be two vectors in $\mathbb{R}^n$. Obviously $d_1(a,b) = 0$ if $a = b$. On the other hand, if $d_1(a,b) = 0$, this means that $|a_1 - b_1| + \cdots + |a_n - b_n| = 0$. Since any modulus is non-negative, we conclude that both $|a_i - b_i| = 0$ for all i, and thus $a = b$.

   For M2, simply notice that

$$\begin{aligned} d_1(a,b) &= |a_1 - b_1| + \cdots + |a_n - b_n| \\ &= |b_1 - a_1| + \cdots + |b_n - a_n| \\ &= d_1(b,a). \end{aligned}$$

   Finally we prove M3. Let $c = (c_1, \dots, c_n)$ be a third element of $\mathbb{R}^n$. Then

$$\begin{aligned} d_1(a,c) &= \sum |a_i - c_i| \\ &= \sum |(a_i - b_i) + (b_i - c_i)| \\ &\leqslant \sum |a_i - b_i| + |b_i - c_i| \\ &= \sum |a_i - b_i| + \sum |b_i - c_i| \\ &= d_1(a,b) + d_1(b,c). \end{aligned}$$

8 Let X be a non-empty set. The discrete metric d on X is given by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise} \end{cases}$$

It is clear from this definition that d satisfies M1 and M2.

It remains to verify the triangle inequality M3. Let $x, y, z \in X$. We want to show that $d(x,y) \leqslant d(x,z) + d(z,y)$. If $d(x,y) = 0$ then the inequality obviously holds. Therefore we assume that $d(x,y) = 1$ so that $x \neq y$. It follows that z must be different from one of x or y so that $d(x,z) = 1$ or $d(z,y) = 1$ or both. Hence

$$d(x,z) + d(z,y) \geqslant 1 = d(x,y).$$