MAS331: Metric Spaces
Solutions to Problems on Chapter 7

1. (a) Here K is not bounded because for any $D \geq 0$, $(0,0),(D+1,0) \in K$ and $d((0,0),(D+1,0)) = D+1 > D$. So K is not compact.

   (b) Here K is not closed because the sequence $(0, \sqrt{1-\frac{1}{n}})$ has all its terms in K but its limit $(0,1)$ is not in K. So K is not compact.

   (c) Here K is not bounded because for any $D \geq 0$, $(0,0),(D+1,0) \in K$ and $d((0,0),(D+1,0)) = D+1 > D$. So K is not compact.

   (d) Clearly K is bounded with $D = 1$. Let (x_n, y_n) be a sequence in K converging to some $(x,y) \in \mathbb{R}^2$. Then each $x_n^2 + y_n^2 = 1$. But $x_n \to x$ and $y_n \to y$, by 2.9, so, by the algebra of limits, $1 = x_n^2 + y_n^2 \to x^2 + y^2$. Hence $x^2 + y^2 = 1$ so $(x,y) \in K$. Therefore K is closed and bounded so it is compact by Heine-Borel.

2. The points in A that are furthest apart are those, such as $(6,4)$ and $(2,4)$, at opposite ends of a diameter of the circle with centre $(4,4)$ and radius 2. So $D = 4$.

   If $(x,y) \in A$ then $d((x,y),(0,0)) \leqslant d((x,y),(4,4)) + d((4,4),(0,0))) \leqslant 2+4\sqrt{2}$. So $M \leqslant 2+4\sqrt{2}$. Also $(4+\sqrt{2}, 4+\sqrt{2}) \in A$ and $d((4+\sqrt{2}, 4+\sqrt{2}),(0,0)) = \sqrt{36+16\sqrt{2}}$. So $M \geqslant \sqrt{36+16\sqrt{2}}$. As $(2+4\sqrt{2})^2 = 4+32+16\sqrt{2} = 36+16\sqrt{2}$, $M = 2+4\sqrt{2}$.

3. Let us suppose that A is compact under d_1 and let (a_n) be a sequence in A. Then (a_n) has a subsequence (a_{n_k}) converging to an element $a \in A$ under d_1. By Problem 3 of Chapter 2, (a_{n_k}) converges to $a \in A$ under d_2 so A is compact under d_2. So if A is compact under d_1 it is compact under d_2. The converse follows by the same argument, interchanging d_1 and d_2. So if A is compact under d_1 it is compact under d_2. So A is compact under d_1 if and only if it is compact under d_2.

   The same result with d_∞ replacing d_1 follows using Problem 4(i) of Assignment 2 in place of Problem 3 of Chapter 2. So A is compact under d_∞ if and only if it is compact under d_2 if and only if it is compact under d_1.

   An alternative approach is to use Heine-Borel and show that A is closed and bounded under all three metrics if it is closed and bounded under any one of them.

4. Let $K_1, \ldots, K_n$ be compact subsets of (X,d) and let $K = K_1 \cup \ldots \cup K_n$. To show that K is compact, let (x_n) be a sequence in K. The terms of the sequence are distributed amongst the n sets $K_1, \ldots, K_n$ so at least one of them, say K_1, therefore must contain infinitely many of the terms. We deduce that (x_n) has a subsequence (x_{n_k}) all of whose of terms are in K_1. As K_1 is compact, the sequence (x_{n_k}) has a subsequence $(x_{n_{k_j}})$

which converges in K_1. Thus the subsequence $(x_{n_{k_j}})$ of (x_n) converges in K (since it already converges in K_1).

5. Suppose first that X is a finite set. Then any sequence in X contains an infinite number of terms which are all equal. In other words, any sequence in X has a subsequence which is constant, and obviously constant (sub)sequences are convergent.

   Conversely suppose X with the discrete metric is compact. If X has infinitely many elements then we can find $x_1, x_2, x_3, \ldots$ in X such that $x_i \neq x_j$ for $i \neq j$ so $(d(x_i, x_j) = 1$ whenever $i \neq j$. Hence (x_n) cannot have a Cauchy subsequence and, as convergent sequences are always Cauchy, (x_n) cannot have a convergent subsequence. This is a contradiction to the compactness of X so X is finite.

6. We adapt the argument from the proof of 7.6 with x replacing 0.

   If $d(a, x) \leqslant M$ for all $a \in A$ then, for all $a, b \in A$,

$$d(a, b) \leq d(a, x) + d(x, b) \leq 2M$$

   so A is bounded with bound $D = 2M$.

   Suppose A is bounded with bound $D = 2M$, fix $b \in A$ and let $m = d(b, x)$. Then for all $a \in A$,

$$d(a, x) \leqslant d(a, b) + d(b, x) \leqslant D + m$$

   so, if $M = D + m$, $d(a, x) \leqslant M$ for all $a \in A$.

7. Pick $x_1 \in F_1, x_2 \in F_2$, and so on. As X is compact, we can find a convergent subsequence $x_{n_k} \to x$ for some $x \in X$. We shall show that $x \in F_1 \cap F_2 \cap \ldots$.

   Let n be a given positive integer. There exists N such that $n_k \geq n$ for all $k \geq N$. Thus $x_{n_k} \in F_{n_k} \subset F_n$ for all $k \geq N$. Now F_n is closed and the subsequence $(x_{n_k} : k \geq N)$ converges to x and has all its terms in F_n. Hence $x \in F_n$, and as n was arbitrary we conclude $x \in F_1 \cap F_2 \cap \ldots$.

8. Consider the function $F : X \to \mathbb{R}$ given by $F(x) = d(f(x), x)$. If $x_n \to x$, then $f(x_n) \to f(x)$ and therefore $d(f(x_n), x_n) \to d(f(x), x)$. Thus F is continuous. As X is compact and F is real-valued, we can conclude from Corollary 7.11 that there exists $x_0 \in X$ such that $F(x_0) = \inf\{F(x) : x \in X\}$. Since x_0 is not a fixed point, $f(x_0) \neq x_0$) so $0 < d(f(x_0), x_0) = F(x_0)$. As $F(x_0) = \inf\{F(x) : x \in X\}$, we have $d(f(x), x) = F(x) \geq F(x_0)$ for all $x \in X$ so we can take $\epsilon = F(x_0)$.

9. With $F_1 = f(X), F_2 = f(F_1)$ and so on, we have that $F_1 \supset F_2 \supset F_3 \supset \ldots$. Since X is compact, $f(X) = F_1$ is compact, $F_2 = f(F_1)$ is compact etc. Thus each F_n is closed as it is compact. By Problem 7, the intersection

$A := F_1 \cap F_2 \cap \ldots$ is non-empty, and A is closed as it is the intersection of closed sets. Finally,

$$
\begin{aligned}
f(A) &= f(F_1 \cap F_2 \cap \ldots) \\
&= f(X \cap F_1 \cap F_2 \cap \ldots) \\
&= f(X) \cap f(F_1) \cap f(F_2) \ldots \\
&= F_1 \cap F_2 \cap F_3 \ldots \\
&= A.
\end{aligned}
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