MAS331: Metric Spaces 2016–17
Solutions to Week 3 Problems on Chapter 2

1. Taking the hint and simplifying we get the following, provided $n > 1$:

$$\begin{aligned} \sqrt{n+1}-\sqrt{n-1} &= \frac{\sqrt{n+1}-\sqrt{n-1}}{1} \\ &= \frac{(\sqrt{n+1}-\sqrt{n-1})(\sqrt{n+1}+\sqrt{n-1})}{\sqrt{n+1}+\sqrt{n-1}} \\ &= \frac{\sqrt{n+1}^2-\sqrt{n-1}^2}{\sqrt{n+1}+\sqrt{n-1}} \\ &= \frac{2}{\sqrt{n+1}+\sqrt{n-1}} \\ &< \frac{2}{2\sqrt{n-1}} = \frac{1}{\sqrt{n-1}}. \end{aligned}$$

As $\frac{1}{\sqrt{n}} \to 0$ we also have $\frac{1}{\sqrt{n-1}} \to 0$. As $\frac{1}{\sqrt{n-1}} \to 0$ it follows from the Sandwich Rule that $d(\sqrt{n+1}, \sqrt{n-1}) \to 0$.

2.

$$\begin{aligned} d_1\left(\left(\frac{n}{n+1}, \frac{n+1}{n}\right), (1,1)\right) &= \left|\frac{n}{n+1}-1\right| + \left|\frac{n+1}{n}-1\right| \\ &= \frac{1}{n+1} + \frac{1}{n} < \frac{2}{n}. \end{aligned}$$

We know that the sequence $\frac{1}{n} \to 0$. By the algebra of limits, $\frac{2}{n} \to 0$ so, by the Sandwich Rule, $d_1\left(\left(\frac{n}{n+1}, \frac{n+1}{n}\right), (1,1)\right) \to 0$. Hence $(\frac{n}{n+1}, \frac{n+1}{n}) \to (1,1)$.

3. Suppose that $a_n \to a$ under d_1. Let $\varepsilon > 0$. Then there exists N such that $a_n \in B_1(a,\varepsilon)$ whenever $n > N$. Hence $a_n \in B_\infty(a,\varepsilon)$ whenever $n > N$ and so $a_n \to a$ under d_∞.

   Conversely, suppose that $a_n \to a$ under d_∞. Let $\varepsilon > 0$. Then $\varepsilon/m > 0$ so there exists N such that $a_n \in B_\infty(a,\varepsilon/m)$ whenever $n > N$. Hence $a_n \in B_1(a,\varepsilon)$ whenever $n > N$ and so $a_n \to a$ under d_1.

   Thus $a_n \to a$ under d_1 if and only if $a_n \to a$ under d_∞.