MAS331: Metric Spaces 2016–17
Solutions to Week 4 Problems on Chapter 2

4. Using d_1, we compute the distance between the nth term and $(0,0)$:

$$\begin{aligned} d_1\left(\left((\sin(\tfrac{1}{n^2+1}), \sqrt{\tfrac{1}{n^4+n}}\right), (0,0)\right) &= |\sin(\tfrac{1}{n^2+1}) - 0| + \left|\sqrt{\tfrac{1}{n^4+n}} - 0\right| \\ &= |\sin(\tfrac{1}{n^2+1})| + \left|\sqrt{\tfrac{1}{n^4+n}}\right| \\ &\leqslant |\tfrac{1}{n^2+1}| + |\tfrac{1}{\sqrt{n^4+n}}| \\ &\leqslant |\tfrac{1}{n^2}| + |\tfrac{1}{n^2}| \\ &= \frac{2}{n^2}. \end{aligned}$$

As $\frac{1}{n^2} \to 0$ it follows from the algebra of limits and the Sandwich Rule that $\frac{2}{n^2} \to 0$ and $d_1\left(\left((\sin(\frac{1}{n^2+1}), \sqrt{\frac{1}{n^4+n}}\right), (0,0)\right) \to 0$. Thus $\left((\sin(\frac{1}{n^2+1}), \sqrt{\frac{1}{n^4+n}}\right) \to (0,0)$.

For d_2, the distance is given by

$$\begin{aligned} d_2\left(\left((\sin(\tfrac{1}{n^2+1}), \sqrt{\tfrac{1}{n^4+n}}\right), (0,0)\right) &= \sqrt{(\sin(\tfrac{1}{n^2+1}) - 0)^2 + \left(\sqrt{\tfrac{1}{n^4+n}} - 0\right)^2} \\ &\leqslant \sqrt{(\tfrac{1}{n^2})^2 + (\tfrac{1}{n^2})^2} \\ &= \sqrt{\frac{2}{n^4}} \\ &= \frac{\sqrt{2}}{n^2}. \end{aligned}$$

As $\frac{1}{n^2} \to 0$ it follows from the algebra of limits and the Sandwich Rule that $\frac{\sqrt{2}}{n^2} \to 0$ and $d_2\left(\left((\sin(\frac{1}{n^2+1}), \sqrt{\frac{1}{n^4+n}}\right), (0,0)\right) \to 0$. Thus $\left((\sin(\frac{1}{n^2+1}), \sqrt{\frac{1}{n^4+n}}\right) \to (0,0)$.

Finally, for d_∞, we have a similar calculation:

$$\begin{aligned} d_\infty\left(\left((\sin(\tfrac{1}{n^2+1}), \sqrt{\tfrac{1}{n^4+n}}\right), (0,0)\right) &= \max\left(\left|\sin(\tfrac{1}{n^2+1}) - 0\right|, \left|\sqrt{\tfrac{1}{n^4+n}} - 0\right|\right) \\ &\leqslant \max\left(\frac{1}{n^2}, \frac{1}{n^2}\right) \\ &= \frac{1}{n^2}, \end{aligned}$$

As $\frac{1}{n^2} \to 0$ it follows from the Sandwich Rule that $d_\infty\left(\left((\sin(\frac{1}{n^2+1}), \sqrt{\frac{1}{n^4+n}}\right), (0,0)\right) \to 0$. Thus $\left((\sin(\frac{1}{n^2+1}), \sqrt{\frac{1}{n^4+n}}\right) \to (0,0)$.

Note that, by Problem **??** and its counterpart on Assignment 2, we only need to do one of these calculations.

5. $|f_n(x) - f(x)| = |\frac{3x}{n} + \frac{2}{n^2}| = \frac{3x}{n} + \frac{2}{n^2}$ as $\frac{3x}{n} + \frac{2}{n^2} > 0$ on $[0,1]$. Also $\frac{3x}{n} + \frac{2}{n^2}$ has positive derivative $\frac{3}{n}$ so it is increasing on $[0,1]$ and takes its maximum value at $x = 1$. Thus $d_\infty(f_n, f) = \frac{3}{n} + \frac{2}{n^2}$ which, by the algebra of limits, tends to 0. Hence (f_n) converges to the function $f(x) = x^2$ in the d_∞ metric. By Proposition 2.15, (f_n) converges to the function $f(x) = x^2$ in the d_1 metric. Alternatively

$$d_1(f_n, f) = \int_0^1 \frac{3x}{n} + \frac{2}{n^2} dx = \frac{3}{2n} + \frac{2}{n^2} \to 0.$$

6. We begin with the d_1-metric. Let's start by computing $d_1(f_n, f)$:

$$\begin{aligned} d_1(f_n, f) &= \int_0^1 |f(x) - f_n(x)| dx \\ &= \int_0^1 1 - \frac{n}{n+x} dx \\ &= 1 - n \int_0^1 \frac{1}{n+x} dx \\ &= 1 - n(\ln(n+1) - \ln(n)) \\ &= 1 - \ln\left(\left(1 + \frac{1}{n}\right)^n\right). \end{aligned}$$

We must show that $d_1(f_n, f) \to 0$ as $n \to \infty$. But note that, as $n \to \infty$, $(1 + \frac{1}{n})^n \to e$, so that $\ln\left((1 + \frac{1}{n})^n\right) \to 1$, and hence $d_1(f_n, f) = 1 - \ln\left((1 + \frac{1}{n})^n\right) \to 0$.

Now let's take the d_∞ metric, and as usual start by computing $d_\infty(f_n, f)$.

$$\begin{aligned} d_\infty(f_n, f) &= \sup\left\{\left|\frac{n}{n+x} - 1\right| \, : \, x \in [0,1]\right\} \\ &= \sup\left\{\left|\frac{x}{n+x}\right| \, : \, x \in [0,1]\right\}. \end{aligned}$$

Notice that for $0 \leqslant x \leqslant 1$, we have that $0 \leqslant \frac{x}{n+x} \leqslant \frac{1}{n+x} \leqslant \frac{1}{n}$. It follows that $0 \leqslant d_\infty(f_n, f) \leqslant \frac{1}{n}$. Since $\frac{1}{n} \to 0$ as $n \to \infty$, the Sandwich Rule tells us that $d_\infty(f_n, f) \to 0$ as $n \to \infty$, and this means that $f_n \to f$ as required.

Note that it is enough to show convergence under d_∞ and then apply Proposition 2.15.

7. (a) We find

$$0 \leqslant d(x_n, y_n) \leqslant d(x_n, a) + d(a, y_n) \to 0 + 0 = 0.$$

By the Sandwich Rule, $d(x_n, y_n) \to 0$.

(b) Using the same method,

$$0 \leqslant d(y_n, a) \leqslant d(y_n, x_n) + d(x_n, a) \to 0 + 0 = 0.$$

Hence $d(y_n, a) \to 0$, that is, $y_n \to a$.

(c) By the given identity, we have $|d(x_n, y_n) - d(a, b)| \leqslant d(x_n, a) + d(y_n, b)$, and so

$$|d(x_n, y_n) - d(a, b)| \to 0 + 0$$

as $n \to \infty$. Hence $d(x_n, y_n) \to d(a, b)$.

Here, for completeness, is a proof of the "given identity".

$d(x, z) + d(y, w) + d(z, w) \geqslant d(x, z) + d(y, z) \geqslant d(x, y)$ so

$$d(x, y) - d(z, w) \leqslant d(x, z) + d(y, w).$$

By symmetry,

$$d(z, w) - d(x, y) \leqslant d(x, z) + d(y, w).$$

Combining the two,

$$|d(x, y) - d(z, w)| \leqslant d(x, z) + d(y, w).$$

8. (a) Fix $x \in [0, 1]$. We have to check that $f_n(x) \to 0$. This is clear for $x = 0$ since $f_n(0) = 0$ for all n. If $x > 0$, the question tells you that $f_n(x) = 0$ if $x \geqslant \frac{2}{n}$, which rearranges to $n \geqslant \frac{2}{x}$. So, for $\epsilon > 0$, set $N = \frac{2}{x}$. Then for $n > N$, $|f_n(x) - 0| = 0 < \epsilon$. This means that $f_n(x) \to 0$ as required.

(b) The graph of f_n is a triangle with height 1 and base $\frac{2}{n}$, and so its area is easy to calculate. Using this method we have

$$\int_0^1 f_n(x)\, dx = \frac{1}{2} \times \frac{2}{n} \times 1 = \frac{1}{n}.$$

(c) Using the first part, we find

$$d_1(f_n, 0) = \int_0^1 |f_n(x) - 0|\, dx = \int_0^1 f_n(x)\, dx = \frac{1}{n} \to 0.$$

Thus $f_n \to 0$ in $(C[a, b], d_1)$.

(d) From the graph, it is obvious that the function f_n has a maximum value and that that maximum value is 1. Therefore

$$d_\infty(f_n, 0) = \sup\{|f_n(x) - 0| : 0 \leqslant x \leqslant 1\} = \sup\{f_n(x) : 0 \leqslant x \leqslant 1\} = 1.$$

This is for every n, so $d_\infty(f_n, 0) \not\to 0$. It follows that $f_n \not\to 0$ in $(C[0, 1], d_\infty)$.

(e) Let's take the hint and imagine that we've picked a sequence λ_n and set $g_n(x) = \lambda_n f_n(x)$. Then whatever we choose for the λ_n, the proof that $g_n \to 0$ pointwise is still true! So we have to choose the λ_n to make sure that $g_n \not\to 0$ in the d_1 or d_∞ metrics. Using the same method as before, we can see that $d_1(g_n, 0) = |\lambda_n|/n$, and so if we set $\lambda_n = n$, it follows that $d_1(g_n, 0) = 1$ for all n, so that $g_n \not\to 0$ in d_1. By Proposition 2.15, $g_n \not\to 0$ under d_∞. Alternatively, reusing the old calculation, we find that $d_\infty(g_n, 0) = n$, which does not converge to 0, so that $g_n \not\to 0$ in d_∞.

So $g_n(x) = nf_n(x)$ is the required example.

(f) Suppose that (f_n) does have a limit in $(C[0,1], d_\infty)$. Let us denote it by f. Then since $f_n \to f$ pointwise, we must have $f = 0$. But we already proved that $f_n \not\to 0$, a contradiction.

9. Let X be any non-empty set and let d be the discrete metric.

(a) Let $x \in X$. If $0 < \varepsilon < 1$ then the open ball $B(x, \varepsilon)$ is the singleton $\{x\}$. All other elements of X are at distance 1 from x. If $1 \leq \varepsilon$ then the open ball $B(x, \varepsilon)$ is the whole set X. All elements of X are at distance 1 or 0 from x.

(b) Suppose that (x_n) converges to x in X. Taking $\varepsilon = \frac{1}{2}$, there exists N such that, whenever $n > N$, $x_n \in B(x, \frac{1}{2})$. As $B(x, \frac{1}{2}) = \{x\}$, we have $x_n = x$ whenever $n > N$.

Conversely suppose that there exists N such that for all $n > N$, $x_n = x$. For every $\varepsilon > 0$, $x_n = x \in B(x, \varepsilon)$ so $x_n \to x$; the same N works for all ε.