MAS331: Metric Spaces
Week 5: Solutions to Problems on Chapter 3

1. The examples are not unique.

   (a) $x_n = \frac{1}{200} + \frac{1}{201n} \to \frac{1}{200} \notin (\frac{1}{200}, \frac{1}{100})$. (The constant 201 is chosen so that $x_1 \in F$.)

   (b) $x_n = 1 - \frac{1}{n} \to 1 \notin F = [0,1) \cup (1,2]$.

   (c) $x_n = (\frac{1}{n}, -\frac{1}{n}) \to (0,0) \notin F$.

   (d) $x_n = (1 + \frac{1}{n}, 0) \to (1,0) \notin F$.

2. Write $C = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Take any sequence $((x_n, y_n))$ in C with $(x_n, y_n) \to (a,b)$ (say) in $\mathbb{R}^2$. Then $x_n \to a$ and $y_n \to b$, both in $\mathbb{R}$. Now $(x_n, y_n) \in C$ means that $x_n^2 + y_n^2 = 1$. Letting $n \to \infty$ gives $a^2 + b^2 = 1$, that is $(a,b) \in C$. We conclude from the definition of 'closed' that C is a closed set.

3. Note that A consists of the first quadrant in the plane (not including the axes). A typical open ball of radius r around a point (x,y) is a square centred at (x,y), with sides parallel to the axes, and with side length $2r$.

   Now we prove that A is open. Let $(x,y) \in A$, so that $x, y > 0$, and set $r = \min\{x,y\}$. Then I claim that $B((x,y),r)$ is contained in A, and this will prove that A is open.

   For, let $(a,b) \in B((x,y),r)$, so that $|a-x| < r$ and $|b-y| < r$. But by our choice of r this means that $|a-x| < x$ and $|b-y| < y$. Hence $-x < a - x < x$ and $-y < b - y < y$. Adding x throughout the first and y throughout the second, $0 < a(< 2x)$ and $0 < b(< 2y)$. Thus $(a,b) \in A$.

4. (a) Let (f_n) be a sequence of elements of F that converges to some $f \in C[-1,0]$ under d_∞. Then $f_n(-1) = -1$ for all n. To show that F is closed we must prove that $f \in F$, that is $f(-1) = -1$. By Proposition 2.15, $f_n \to f$ pointwise. In particular, $f_n(-1) \to f(-1)$, and since $f_n(-1) = -1$ for all n this means that $f(-1) = -1$. Thus $f \in F$, as required, and F is closed..

   (b) For $n \geq 1$ and $x \in [-1,0)$, $|f_n(x) - g(x)| = |x^{2n}| = x^{2n}$. Hence

   $$d_1(f_n, g) = \int_{-1}^{0} x^{2n} dx = [x^{2n+1}/(2n+1)]_{-1}^{0} = 1/(2n+1) < 1/n.$$

   As $1/n \to 0$, $d_1(f_n, g) \to 0$ by the Sandwich Rule. Therefore $f_n \to g$ under d_1. But each $f_n \in F$, because $(-1)^{2n} - 2 = -1$, and $g \notin F$ because $g(-1) = -2$. Therefore F is not closed under d_1.

   (c) If (f_n) is convergent to f in $(C[-1,0], d_\infty)$ then, Proposition 2.15, it is convergent to f in $(C[-1,0], d_1)$. Therefore $f = g$. But F is closed under d_∞ so $f \in F$, which is false as $g \notin F$. Therefore (f_n) is not convergent in $(C[-1,0], d_\infty)$.

5. Take the sequence $q_1 = 1, q_2 = 1.4, q_3 = 1.41, q_4 = 1.414, \cdots$. Thus q_n is the decimal expansion of $\sqrt{2}$ up to $n-1$ decimal places and $q_n \to \sqrt{2}$ in $\mathbb{R}$. Each $\sqrt{2} - q_n \in \mathbb{I}$ because if $\sqrt{2} - q_n \in \mathbb{Q}$ then $\sqrt{2} = (\sqrt{2} - q_n) + q_n \in \mathbb{Q}$. By the algebra of limits, $\sqrt{2} - q_n \to 0 \notin \mathbb{I}$. Therefore $\mathbb{I}$ is not closed in $\mathbb{R}$.

6. Write $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leqslant 5\}$. This is the closed ball $B[(0,0), \sqrt{5}]$ so it is closed by 3.3.

   Alternatively, take any sequence $((x_n, y_n))$ in D with $(x_n, y_n) \to (a, b)$ (say) in $\mathbb{R}^2$. Then $x_n \to a$ and $y_n \to b$, both in $\mathbb{R}$. Now $(x_n, y_n) \in D$ means that $x_n^2 + y_n^2 \leqslant 5$. Letting $n \to \infty$ and applying 3.2 to the sequence $5 - x_n^2 - y_n^2$ gives $a^2 + b^2 \leqslant 5$, that is $(a, b) \in D$. We conclude from the definition of 'closed' that D is a closed set.

7. $d_2((a_1 + r(1 - \frac{1}{n}), a_2, \ldots, a_m), (a_1, a_2 \ldots, a_m)) = r(1 - \frac{1}{n}) < r$ so each $(a_1 + r(1 - \frac{1}{n}), a_2, \ldots, a_m) \in B$. Also $d_2((a_1 + r(1 - \frac{1}{n}), a_2, \ldots, a_m), (a_1 + r, a_2, \ldots, a_m)) = \frac{r}{n} \to 0$ so $(a_1 + r(1 - \frac{1}{n}), a_2, \ldots, a_m) \to (a_1 + r, a_2, \ldots, a_m)$. But $d_2((a_1 + r, a_2, \ldots, a_m), (a_1, a_2, \ldots, a_m)) = r$ so $(a_1 + r, a_2, \ldots, a_m) \notin B$. Therefore B is not closed.

8. (a) Suppose that U is closed under d_1 and let (u_n) be a sequence in U converging, under d_∞, to $x \in X$. Then, by Problem 3 of Chapter 2, $(u_n) \to x$ under d_1. As U is closed under d_1, $x \in U$ so U is closed under d_∞.

   Conversely suppose that U is closed under d_∞ and let (u_n) be a sequence in U converging, under d_1, to $x \in X$. Then, by Problem 3 of Chapter 2, $(u_n) \to x$ under d_∞. As U is closed under d_∞, $x \in U$ so U is closed under d_1.

   (b) Suppose that U is open under d_∞. There exists $r > 0$ such that $B_\infty(a, r) \subseteq U$. But, by Problem 5 of Chapter 1, $B_1(a, r) \subseteq B_\infty(a, r)$ so $B_1(a, r) \subseteq U$ and therefore U is open under d_1.

   Conversely suppose that U is open under d_1. There exists $r > 0$ such that $B_\infty(a, r) \subseteq U$. But, by Problem 5 of Chapter 1, $B_\infty(a, r/m) \subseteq B_1(a, r)$ so $B_\infty(a, r/m) \subseteq U$ and therefore U is open under d_∞.

   (c) Proof of (b) using (a). U is open under d_1 if and only if $X \backslash U$ is closed under d_1 (by Proposition 3.15) if and only if $X \backslash U$ is closed under d_∞ (by (a)) if and only if U is open under d_∞ (by Proposition 3.15).

   Proof of (a) using (b). U is closed under d_1 if and only if $X \backslash U$ is open under d_1 (by Proposition 3.15) if and only if $X \backslash U$ is open under d_∞ (by (b)) if and only if U is closed under d_∞ (by Proposition 3.15).

9. A set consisting of a single point a is closed Any sequence (x_n) in $\{x\}$ must be constant: $x_n = x$ for all n, and its limit must be x which is in $\{x\}$. Therefore $\{x\}$ is closed. A finite set $\{x_1, x_2, \ldots, x_n\}$ can be written $\{x_1\} \cup \{x_2\} \cup \ldots \cup \{x_n\}$, which is closed, by 3.18, because it is the union of a finite number of closed sets.

10. $(x, y) \in A$ if and only if $1 \leqslant x^2+y^2$ and $x^2+y^2 \leqslant 4$. Now $1 \leqslant x^2+y^2$ if and only if $(x, y) \notin B((0,0), 1)$ and $x^2+y^2 \leqslant 4$ if and only if $(x, y) \in B[(0,0), 2]$. Hence $A = B[(0,0), 2] \cap (\mathbb{R}^2 \backslash B((0,0), 1)$. Also $B[(0,0), 2]$ is closed and, as $B((0,0), 1)$ is open, $(\mathbb{R}^2 \backslash B((0,0), 1)$ is closed by 3.15. By 3.18, A is closed.

    Let $C = \{(x, y) \in \mathbb{R} : 1 < x^2+y^2 < 4\}$. $(x, y) \in C$ if and only if $1 < x^2+y^2$ and $x^2 + y^2 < 4$. Now $1 < x^2 + y^2$ if and only if $(x, y) \notin B[(0,0), 1]$ and $x^2 + y^2 < 4$ if and only if $(x, y) \in B((0,0), 2)$. Hence $C = B((0,0), 2) \cap (\mathbb{R}^2 \backslash B[(0,0), 1]$. Also $B((0,0), 2)$ is open and, as $B[(0,0), 1]$ is closed, $\mathbb{R}^2 \backslash B[(0,0), 1]$ is open by 3.15. By 3.17, C is open.

11. Recall that if $y \neq x$, its distance from x is defined to be 1. It follows that the only element which is a distance of less than $\frac{1}{2}$, say, from x, is x itself. That is,

    $$B(x, \tfrac{1}{2}) = \{y \in X : d(y, x) < \tfrac{1}{2}\} = \{x\},$$

    and similarly,

    $$B[x, \tfrac{1}{2}] = \{y \in X : d(y, x) \leqslant \tfrac{1}{2}\} = \{x\}.$$

    Every subset A of X is then open because, for each $a \in A$, it is clear that A contains the open ball $B(x, \frac{1}{2})$. Every subset A of X is closed, by 3.15, because its complement $X \backslash A$ is open.