MAS331: Metric Spaces 2016-17
Solutions to Problems on Chapter 4

1. (a) Let $(x, y) \in \mathbb{R}^2$. Take a sequence (x_n, y_n) in $\mathbb{R}^2$ tending to (x, y). Then, by the d_2-version of Prop.2.9, $x_n \to x$ and $y_n \to y$. By the algebra of limits, $x_n y_n \to xy$, or, in other words, $f(x_n, y_n) \to f(x, y)$. Therefore f is continuous at (x, y), and therefore (since (x, y) was arbitrary) at all points of $\mathbb{R}^2$.

   (b) By definition, $f^{-1}[1, \infty) = \{(x, y) \in \mathbb{R}^2 \mid xy \geqslant 1\}$. As $[1, \infty)$ is closed in $\mathbb{R}$, and f is continuous, $f^{-1}[1, \infty)$ is closed in $\mathbb{R}^2$.

2.
   - f is continuous at any point $(x, y) \neq (0, 0)$. To see this, let $(x_n, y_n) \to (x, y)$, so that $x_n \to x$, $y_n \to y$, by the d_2-version of Prop.2.9, and the algebra of limits gives that $\frac{x_n y_n}{x_n^2 + y_n^2} \to \frac{xy}{x^2+y^2}$, since $x^2 + y^2 \neq 0$. By 4.2, f is continuous at (x, y).
   - Let $k \in \mathbb{R}$. Then $((\frac{1}{n}, \frac{k}{n})) \to (0, 0)$ but $f_c(\frac{1}{n}, \frac{k}{n}) = \frac{k}{1+k^2}$. If f_c is continuous then $f_c(\frac{1}{n}, \frac{k}{n}) \to f_c(0, 0) = c$ so $c = \frac{k}{1+k^2}$. Taking $k = 1$, we get $c = \frac{1}{2}$ and taking $k = 2$, we get $c = \frac{2}{5}$, a contradiction. So f_c is not continuous at $(0, 0)$.

3. Let $x_n \to x$ in X. By Def.4.2, we need to show that $f(x_n) \to f(x)$ in Y. This follows by the Sandwich Rule since

$$d_Y(f(x_n), f(x)) \leqslant \lambda d_X(x_n, x) \to \lambda \times 0 = 0.$$

4. Problem 3 of Chapter 2 tells us that a sequence $x_n \to x$ under d_1 if and only if $x_n \to x$ under d_∞. (a)⇒(b): Suppose that $f : (\mathbb{R}^m, d_1) \to (\mathbb{R}^m, d_1)$ is continuous. Let $(x_n) \to x$ in $(\mathbb{R}^m, d_1)$. Then $(f(x_n)) \to f(x)$ in $(\mathbb{R}^m, d_1)$. By Problem 3 of Chapter 2, $(f(x_n)) \to f(x)$ in $(\mathbb{R}^m, d_\infty)$. Therefore $f : (\mathbb{R}^m, d_1) \to (\mathbb{R}^m, d_\infty)$ is continuous.

   (b)⇒(c): Suppose that $f : (\mathbb{R}^m, d_1) \to (\mathbb{R}^m, d_\infty)$ is continuous. Let $(x_n) \to x$ in $(\mathbb{R}^m, d_\infty)$. By Problem 3 of Chapter 2, $(x_n) \to x$ in $(\mathbb{R}^m, d_1)$. Then $(f(x_n)) \to f(x)$ in $(\mathbb{R}^m, d_\infty)$. Therefore $f : (\mathbb{R}^m, d_1) \to (\mathbb{R}^m, d_\infty)$ is continuous.

   (c)⇒(d): Suppose that $f : (\mathbb{R}^m, d_\infty) \to (\mathbb{R}^m, d_\infty)$ is continuous. Let $(x_n) \to x$ in $(\mathbb{R}^m, d_\infty)$. Then $(f(x_n)) \to f(x)$ in $(\mathbb{R}^m, d_\infty)$. By Problem 3 of Chapter 2, $(f(x_n)) \to f(x)$ in $(\mathbb{R}^m, d_1)$. Therefore $f : (\mathbb{R}^m, d_\infty) \to (\mathbb{R}^m, d_1)$ is continuous.

   (d)⇒(a): Suppose that $f : (\mathbb{R}^m, d_\infty) \to (\mathbb{R}^m, d_1)$ is continuous. Let $(x_n) \to x$ in $(\mathbb{R}^m, d_1)$. By Problem 3 of Chapter 2, $(x_n) \to x$ in $(\mathbb{R}^m, d_\infty)$. Then $(f(x_n)) \to f(x)$ in $(\mathbb{R}^m, d_1)$. Therefore $f : (\mathbb{R}^m, d_\infty) \to (\mathbb{R}^m, d_1)$ is continuous.

5. Suppose that f is continuous when $C[0, 1]$ has the metric d_1. Let (x_n) be a sequence in $C[0, 1]$ converging to $x \in C[0, 1]$ under the metric d_∞.

Then, by Proposition 2.15, (x_n) converges to x under d_1. By continuity of f under d_1, $f(x_n)$ converges to $f(x)$ in X. Therefore f is continuous when $C[0,1]$ has the metric d_∞.

Now suppose that g is continuous when $C[0,1]$ has the metric d_∞. Let (x_n) be a sequence in X converging to $x \in X$. By continuity of g under the metric d_∞, $(g(x_n))$ converges to $g(x)$ in $C[0,1]$ under the metric d_∞. By Proposition 2.15, $(g(x_n))$ converges to $g(x)$ in $C[0,1]$ under the metric d_1. Therefore g is continuous when $C[0,1]$ has the metric d_1.

6. This is on Assignment 3.

7. (a) Let $(x_n) \to x$ in X. For each n, $f(x_n) = y$ and also $f(x) = y$. Clearly $(f(x_n)) \to f(x)$ in Y so f is continuous.

   (b) We use Def.4.2. Let $x_n \to x$ in X. As f is continuous, we must have $f(x_n) \to f(x)$ in Y. As g is continuous we must have $g(f(x_n)) \to g(f(x))$ as required.

   (c) Now we use Def.4.1. Let $x \in X$ and let $\varepsilon > 0$. Applying 4.1 to g, ε and $f(x)$, there exists $\rho > 0$ such that $g(B(f(x), \rho)) \subseteq B(g(f(x), \varepsilon)$. Applying Def.4.1 to f, ρ and x, there exists $\delta > 0$ such that $f(B(x, \delta)) \subseteq B(f(x), \rho)$. Combining the two, if $b \in B(x, \delta)$ then $f(b) \in B(f(x), \rho)$ and $g(f(b)) \in B(g(f(x), \varepsilon)$. Hence $g \circ f(B(x, \delta)) \subseteq B(g(f(x), \varepsilon)$. By 4.1, the composite $g \circ f \colon X \to Z$ is continuous.

8. Let $A = \{0\}$ (any closed subset of $\mathbb{R}$ containing 0 but not containing 1 will do; A is closed by Problem 9 of Chapter 3). Then $f^{-1}(A)$ consists of all real numbers x such that $f(x) \in A$, i.e., $f(x) = 0$. This set is exactly $\mathbb{R} \setminus \{0\}$, which is not closed in $\mathbb{R}$ because it contains the terms, but not the limit of the convergent sequence $(\frac{1}{n})$. Consequently, by Theorem 4.13, f is not continuous .

9. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = x^2 + y^2$. Then f is a continuous function from $(\mathbb{R}^2, d_2)$ to $\mathbb{R}$. The method, using the sequential definition of continuity and the algebra of limits, is the same as for $x^2 - y^3$ in the notes or xy in Problem 1.

   $A = \{(x, y) \in \mathbb{R} : 1 \leqslant x^2 + y^2 \leqslant 4\} = f^{-1}([1, 4])$ and $C = \{(x, y) \in \mathbb{R} : 1 < x^2 + y^2 < 4\} = f^{-1}((1, 4))$. As $[1, 4]$ is closed in $\mathbb{R}$, A is closed in $\mathbb{R}^2$ by 4.9. As $(1, 4)$ is open in $\mathbb{R}$, C is open in $\mathbb{R}^2$ by Theorem 4.13.