MAS331: Metric Spaces 2015-16
Solutions to Problems on Chapter 5

1. (a) Let $m > n$. Then $f_m(x) - f_n(x) = \frac{x^{n+1}}{2^{n+1}} + \cdots + \frac{x^m}{2^m} \geqslant 0$ for all $x \in [0,1]$. For $x \in (0,1]$, $\frac{d}{dx}(f_m(x) - f_n(x)) = (n+1)\frac{x^n}{2^{n+1}} + \cdots + m\frac{x^{m-1}}{2^m} > 0$ so $f_m - f_n$ is strictly increasing on $(0,1]$ and takes its maximum value at 1. Then

$$\begin{aligned} d_\infty(f_n, f_m) &= f_m(1) - f_n(1) \\ &= \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^m} \\ &= \frac{1}{2^{n+1}}\left(\frac{1 - (1/2)^{m-n}}{1/2}\right) \text{ by the formula below} \\ &= \frac{1}{2^n}\left(1 - \frac{1}{2^{m-n}}\right) \\ &= \frac{1}{2^n} - \frac{1}{2^m}. \end{aligned}$$

The formula referred to is the formula for the sum of a finite geometric progression with initial term a and common ration r:

$$a + ar + \cdots + ar^k = \frac{a(1 - r^{k+1})}{1 - r}, r \neq 1$$

It is applied here with $a = \frac{1}{2^{n+1}}$, $r = \frac{1}{2}$ and $k = m - n - 1$.

(b) The sequence $(\frac{1}{2^n})$ is convergent (to 0) in $\mathbb{R}$ and so it is a Cauchy sequence in $\mathbb{R}$. Therefore, given any $\epsilon > 0$ there exists N such that $|\frac{1}{2^n} - \frac{1}{2^m}| < \epsilon$, whenever $m, n > N$. That is $d_\infty(f_n, f_m) < \epsilon$ whenever $m, n > N$. Thus (f_n) is a Cauchy sequence.

(c) Yes, (f_n) does converge in $(C[0,1], d_\infty)$. This is because it is a Cauchy sequence and we know that the metric space $(C[a,b], d_\infty)$ is complete for all $a < b$, in particular for $a = 0$ and $b = 1$.

You are not asked for the limit which is

$$1 + \frac{x}{2} + \frac{x^2}{2^2} + \cdots + \frac{x^n}{2^n} + \cdots = \frac{2}{2 - x}.$$

2. This will be on Assignment 3.

3. Any Cauchy sequence in $(0, \infty)$ must converge in $\mathbb{R}$ so we want a sequence in $(0, \infty)$ that converges in $\mathbb{R}$ to a limit outside $(0,1)$. As $[0,1]$ is closed that limit must be 0 or 1 so we take $x_n = \frac{1}{n} \in (0, \infty)$. Then (x_n) is Cauchy, because it is convergent in $\mathbb{R}$, but its limit, 0 is not in $(0, \infty)$.

For $\mathbb{I}$, we could take $\sqrt{2} - 1, \sqrt{2} - \frac{14}{10}, \sqrt{2} - \frac{141}{100}, \sqrt{2} - \frac{1414}{1000}, \ldots$ which converges in $\mathbb{R}$ to $0 \notin \mathbb{I}$. It is Cauchy, because it is convergent in $\mathbb{R}$, but its limit, 0 is not in $\mathbb{I}$. Another possibility for $\mathbb{I}$ is $(\frac{\sqrt{2}}{n})$ which converges to 0.

4. (a) As $Jd'(x,y) \leqslant d(x,y)$, we get $d'(x,y) \leqslant J^{-1}d(x,y)$ and as $d(x,y) \leqslant Kd'(x,y)$, we get $K^{-1}d(x,y) \leqslant d'(x,y)$.

   (b) Let $x_n \to x$ in (X,d). Then $d(x_n,x) \to 0$ as $n \to \infty$. But $0 \leqslant d'(x_n,x) \leqslant J^{-1}d(x_n,x)$, so that by the Sandwich Rule $d'(x_n,x) \to 0$ as $n \to \infty$. Therefore $x_n \to x$ in (X,d'). The converse follows in the same way using $d(x,y) \leqslant Kd'(x,y)$.

   (c) Suppose that (x_n) is Cauchy in (X,d). Let $\epsilon > 0$. Then $J\epsilon > 0$ so there exists N such that $d(x_n,x_m) < J\epsilon$ for $n,m > N$. Then $d'(x_n,x_m) \leqslant J^{-1}d(x_n,x_m) < \epsilon$, and so (x_n) is Cauchy in (X,d'). The converse follows in the same way.

   (d) Suppose that (X,d) is complete. We'll show that (X,d') is complete. Let (x_n) be a Cauchy sequence in (X,d'). Then by (c), (x_n) is Cauchy in (X,d), and so converges in (X,d). But then by (b), (x_n) converges in (X,d'). This shows that (X,d') is complete. The converse follows in the same way.

5. By Problem 3 of Chapter 1, $d_\infty(a,b) \leqslant d_2(a,b) \leqslant \sqrt{m}d_\infty(a,b)$ and $d_\infty(a,b) \leqslant d_1(a,b) \leqslant md_\infty(a,b)$ for all $a,b \in \mathbb{R}^m$. By Theorem 5.10, $\mathbb{R}^m$ is complete under d_2. By Problem 4(d), it is complete under d_∞ and,by Problem 4(d) again, it is complete under d_1.

6. We take the same sequence as in Problem 3, $x_n = \frac{1}{n}$, which is Cauchy as it is convergent in $\mathbb{R}$, and take $f(x) = \frac{1}{x}$. Then f is continuous by 4.2 because if (y_n) is a sequence converging in $(0,\infty)$ to some limit $y \in (0,\infty)$ then each $y_n \neq 0$ and $y \neq 0$ so, by the algebra of limits $f(y_n) = \frac{1}{y_n} \to \frac{1}{y} = f(y)$. The sequence $(f(x_n))$ is (n) which is not Cauchy as $d(n,m) = |m-n| \geqslant 1$ when $m \neq n$.

7. Let (x_n) be a Cauchy sequence in X with respect to the discrete metric d. Then there exists N such that for all $m,n > N$, $d(x_n,x_m) < 1$. But for $x,y \in X$, $d(x,y)$ can only take two values 0 or 1. So, for all $m,n > N$, $d(x_n,x_m) = 0$ and $x_n = x_m$. If $a = x_{n'}$ for the smallest integer $n' > N$ then $x_n = a$ for all $n > N$. The sequence (x_n) is then convergent to a because, for all $\varepsilon > 0$ and all $n > N$, $d(x_n,a) = d(a,a) = 0 < \varepsilon$. Hence (X,d) is complete.