MAS331: Metric Spaces 2015-16
Solutions to Problems on Chapter 6

1. (a) See lecture notes.

   (b) $g(1) = 2k \geqslant 1$ and, for $x > 1$, $g'(x) = k(1 - \frac{1}{x^2}) > 0$ (because $x^2 > 1$ and so $\frac{1}{x^2} < 1$). Therefore g is increasing on $(1, \infty)$ and, for $x > 1$, $g(x) > g(1) \geqslant 1$. Hence $g(x) \in [1, \infty)$ for all $x \in g(x) \in [1, \infty)$.

   (c) $[1, \infty)$ is closed in $\mathbb{R}$ and $\mathbb{R}$ is complete. So $[1, \infty)$ is complete (5.8). By the Contraction Mapping Principle, g has a unique fixed point x in $[1, \infty)$. When $k = \frac{3}{4}$, x satisfies $\frac{3}{4}(x + \frac{1}{x}) = x$ so, multiplying through by $4x$, $3x^2 + 3 = 4x^2$ so $x^2 = 3$ and, as $x \in [1, \infty)$, $x = \sqrt{3}$.

   (d) By the remark following the Contraction Mapping Principle, $|x_n - x| \leq \frac{4\times 3^n}{4^n}|x_1 - x_0|$. As $x_0 = 1$ and $x_1 = \frac{3}{2}$, $|x_1 - x_0| = \frac{1}{2}$ and the result follows.

2. Let $(x, y, z), (x', y', z') \in \mathbb{R}^3$. Then

$$\begin{aligned}
d_2(f((x,y,z)),(x',y',z')) &= \sqrt{(\tfrac{1}{2}\cos y - \tfrac{1}{2}\cos y')^2 + (\tfrac{2}{3}\sin z - \tfrac{2}{3}\sin z')^2 + (\tfrac{3}{4}x - \tfrac{3}{4}x')^2} \\
&= \sqrt{\tfrac{1}{4}(\cos y - \cos y')^2 + \tfrac{4}{9}(\sin z - \sin z')^2 + \tfrac{9}{16}(x - x')^2} \\
&\leqslant \sqrt{\tfrac{9}{16}|\cos y - \cos y'|^2 + \tfrac{9}{16}|\sin z - \sin z'|^2 + \tfrac{9}{16}|x - x'|^2} \\
&\leqslant \tfrac{3}{4}\sqrt{|y - y'|^2 + |z - z'|^2 + |x - x'|^2} \\
&= \tfrac{3}{4} d_2((x,y,z),(x',y',z'))
\end{aligned}$$

   As $\frac{3}{4} < 1$, we see that f is a contraction.

3. First note that $|\sin(\frac{x+y}{2})| = \sin(|\frac{x+y}{2}|)$. Also $|\frac{x+y}{2}| \leqslant |\frac{x}{2}| + |\frac{y}{2}| \leqslant 1$ so, as sin is increasing, from 0 to $\sin(1)$, on $[0, 1]$, we have $\sin(|\frac{x+y}{2}|) \leqslant \sin(1)$. As in the notes,

$$\begin{aligned}
|\cos x - \cos y| &= \left|2\sin\left(\tfrac{x-y}{2}\right)\sin\left(\tfrac{x+y}{2}\right)\right| \\
&\leqslant 2\left|\sin\left(\tfrac{x-y}{2}\right)\right|\sin(1) \\
&\leqslant 2\tfrac{|x-y|}{2}\sin(1) \\
&= \sin(1)|x - y|.
\end{aligned}$$

   As $0 < \sin(1) < 1$, cos is a contraction on $[-1, 1]$.

4.
$$|f'(x)| = \left|\frac{3x^2 + 2x}{6}\right| \leqslant \frac{5}{6}$$

   for all $x \in (-1, 1)$. By the Differential Criterion, f is a contraction of $[-1, 1]$. By Proposition 5.8, $[-1, 1]$ is complete, being closed in the complete space $\mathbb{R}$. By the Contraction Mapping Principle, there is a unique value of x satisfying $x = f(x)$, or, rearranging, $x^3 + x^2 - 6x + 1 = 0$, in the interval $[-1, 1]$.

5. On Assignment 6.

6. We use the Differential Criterion. Here, $h'(x) = -e^{-x}$ so $|h'(x)| = e^{-x}$ which is decreasing. So $|h'(x)| = e^{-x} \leqslant e^{-1}$ for all x. Since $1/e < 1$, h is a contraction. As $[1, \infty)$ is closed in $\mathbb{R}$, and therefore complete, the Contraction Mapping Principle says it has a unique fixed point in $[1, \infty)$.

7. On Assignment 6.

8. (a) When $f(x) = \sin x$ we find

$$(T(f))(x) = 1 + \int_0^x \sin u \, du = 1 - [\cos u]_0^x = 1 - \cos x + \cos 0 = 2 - \cos x.$$

(b) We have

$$|f(x) - g(x)| \leqslant \max_{0 \leqslant t \leqslant 1/2} |f(t) - g(t)| \leqslant d_\infty(f, g)$$

so that

$$|(T(f))(x) - (T(g))(x)| = \left| \int_0^x (f(u) - g(u)) \, du \right| \leqslant \int_0^x d_\infty(f, g) \, du = x d_\infty(f, g) \leqslant \frac{1}{2} d_\infty(f, g).$$

This holds for any $x \in [0, \frac{1}{2}]$, so that we can take the maximum of the left hand side to find

$$d_\infty(T(f), T(g)) = \max_{0 \leqslant t \leqslant 1/2} |(T(f))(t) - (T(g))(t)| \leqslant \frac{1}{2} d_\infty(f, g).$$

This says that T is a contraction, and so has a fixed point.

(c) Now take $f_1(x) = 1 + x$ for $x \in [0, \frac{1}{2}]$. Put $f_2 = T(f_1)$, $f_3 = T(f_2)$, $f_4 = T(f_3)$, and so on. Then in fact

$$\begin{aligned} f_2(x) &= 1 + \int_0^x f_1(u) \, du = 1 + \int_0^x (1 + u) \, du = 1 + x + \frac{x^2}{2}, \\ f_3(x) &= 1 + \int_0^x f_2(u) \, du = 1 + \int_0^x (1 + u + \frac{u^2}{2}) \, du = 1 + x + \frac{x^2}{2} + \frac{x^3}{2.3} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}. \end{aligned}$$

Thus it looks as if

$$f_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

(a fact which we could now establish by induction). The limit of the sequence (f_n) should therefore be

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = e^x.$$

(d) If $f(x) = e^x$, we find that

$$1+\int_0^x f_1(u)\,du = 1+\int_0^x e^u\,du = 1+[e^u]_0^x = 1+(e^x-1) = e^x = f(x).$$

Thus e^x does satisfy the given integral equation.

9. (a) Putting $f_1(x) = 1$ into the formula

$$(T(f))(x) = 1 + 3\int_0^x u^2 f(u)\ du.$$

for f we see that

$$f_2(x) = (T(f_1))(x) = 1 + 3\int_0^x u^2.1\,du = 1 + x^3.$$

Then putting f_2 in we find

$$f_3(x) = (T(f_2))(x) = 1 + 3\int_0^x u^2(1+u^3)\,du = 1 + x^3 + \frac{x^6}{2}.$$

And then

$$f_4(x) = (T(f_3))(x) = 1+3\int_0^x u^2(1+u^3+\frac{u^6}{2})\,du = 1+x^3+\frac{x^6}{2}+\frac{x^9}{2.3}.$$

A reasonable guess for f_n is therefore

$$f_n(x) = 1 + x^3 + \frac{x^6}{2!} + \frac{x^9}{3!} + \cdots + \frac{x^{3n}}{n!},$$

and then the limit function appears to be

$$1 + x^3 + \frac{x^6}{2!} + \frac{x^9}{3!} + \cdots + \frac{x^{3n}}{n!} + \cdots = e^{x^3}.$$

(b) Putting $f(x) = e^{x^3}$, we see that

$$1 + 3\int_0^x u^2 f(u)\ du = 1 + \int_0^x 3u^2.e^{u^3}\ du = 1 + (e^{x^3} - 1) = e^{x^3}.$$

(c) It is easy to check that the function f satisfies these properties.

10. Using axioms M3 and M2,

$$\begin{aligned} d(x_m, x_n) &\leqslant d(x_m, x) + d(x, x_n) = d(x_m, x) + d(x_n, x) \text{ and} \\ d(x, x_n) &\leqslant d(x, x_m) + d(x_m, x_n) = d(x_m, x) + d(x_m, x_n). \end{aligned}$$

Rearranging the first of these gives

$$d(x_m, x_n) - d(x, x_n) \leqslant d(x, x_m)$$

and rearranging the second gives

$$-d(x, x_m) \leq d(x_m, x_n) - d(x, x_n)$$

so

$$-d(x, x_m) \leq d(x_m, x_n) - d(x, x_n) \leq d(x, x_m).$$

Now fix n and let $m \to \infty$. Then, as $x_m \to x$, $d(x, x_m) \to 0$ and $-d(x, x_m) \to 0$. By the Sandwich Rule, $d(x_m, x_n) \to d(x, x_n)$.