MAS331 METRIC SPACES *Assignment 2*

1. For $n \geq 1$, let (f_n) be the sequence in $C[0, 2\pi]$ such that $f_n(x) = \frac{\cos(nx)}{n}$ for $n \geqslant 1$ and $0 \leq x \leq 2\pi$. Let g be the zero function in $C[0, 2\pi]$, $g(x) = 0$ for $0 \leq x \leq 2\pi$. Compute $d_\infty(f_n, g)$ and deduce that $(f_n) \to g$ in $(C[0, 2\pi], d_\infty)$.

   Does (f_n) converge in $(C[0, 2\pi], d_1)$? Does (f_n) converge pointwise in $C[0, 2\pi]$? Justify your answers.

   Hint. With the trigonometric functions it is often best to use the facts that $|\cos(y)| \leqslant 1$ and $|\sin(y)| \leqslant 1$ for all y. Using the approach via stationary points isn't always easy; with $\frac{\cos(nx)}{n}$ in the question, the larger n is the more stationary points there are $(0, \frac{\pi}{n}, \frac{2\pi}{n}, \frac{3\pi}{n}, \ldots, \frac{2n\pi}{n})$. Also, if you try to calculate $d_1(f_n, g)$, it is complicated by the number of places where $\cos(nx)$ changes sign. But you don't have to calculate it to answer the question; you can just apply a result from the notes.

2. In the set $C[0, 1]$, let $F = \{f \in C[0, 1] : f(1) = 0\}$.

   (a) Show that F is a closed subset of $(C[0, 1], d_\infty)$.

   (b) For $n \geq 1$, let (f_n) be the sequence in $C[0, 1]$ such that $f_n(x) = 1 - x^n$ for $n \geqslant 1$ and $0 \leq x \leq 1$. Let g be the constant function in $C[0, 1]$ such that $g(x) = 1$ for $0 \leq x \leq 1$. Show that $(f_n) \to g$ in $(C[0, 1], d_1)$. Deduce that F is not a closed subset of $(C[0, 1], d_1)$.

3. In $(\mathbb{R}^2, d_2)$, let

$$\begin{aligned} A_1 &= \{(x, y) \in \mathbb{R}^2 : -1 < y \leqslant 1\}; \\ A_2 &= \{(x, y) \in \mathbb{R}^2 : -1 \leqslant y \leqslant 1\}; \\ A_3 &= \{(x, y) \in \mathbb{R}^2 : -1 < y < 1\}. \end{aligned}$$

   (a) Show that A_1 is neither open nor closed in $\mathbb{R}^2$.

   (b) Show that A_2 is closed in $\mathbb{R}^2$.

   (c) Show that A_3 is open in $\mathbb{R}^2$.

   Hint. When trying to show that a subset of $\mathbb{R}^2$ is open (or not), it helps to draw a diagram to sharpen your intuition. But be warned that a diagram is NOT a proof!

4. This question concerns the metrics d_2 and d_∞ on $\mathbb{R}^m$ and uses the notation $B_2(a, \varepsilon)$ and $B_\infty(a, \varepsilon)$ to distinguish between open balls in the two metrics. From Problem 5 on Chapter 1, we know that, for all $a \in \mathbb{R}^m$ and all $\varepsilon > 0$,

$$B_\infty(a, \varepsilon/\sqrt{m}) \subseteq B_2(a, \varepsilon) \subseteq B_\infty(a, \varepsilon).$$

   Use this to show the following.

   (i) If (a_n) is a sequence in $\mathbb{R}^m$ then $(a_n) \to x$ under d_2 if and only if $(a_n) \to x$ under d_∞.

   (ii) If A is a subset of $\mathbb{R}^m$ then A is closed under d_2 if and only if A is closed under d_∞.

   (iii) If A is a subset of $\mathbb{R}^m$ then A is open under d_2 if and only if A is open under d_∞.

Assignment set on Monday 23rd October, for handing in on Tuesday 31st October.