MAS331 METRIC SPACES *Assignment 3*

1. (Problem 5 on Chapter 6) Use the differential criterion to show that $\cos \circ \sin$ is a contraction on $\mathbb{R}$ with contraction factor $k = \sin(1)$. Deduce that there is a unique element $x \in \mathbb{R}$ such that $\cos(\sin(x)) = x$.

2. (Problem 7 on Chapter 6) Let f and g both be contractions of the metric space (X, d) with contraction factors k and k' respectively. Show that the function $f \circ g$ (which takes x to $f(g(x))$) is a contraction of (X, d) with contraction factor kk'.

Show also that if x is a fixed point of $f \circ g$ then $g(f(g(x))) = g(x)$. Hence find a fixed point of $g \circ f$ (in terms of x).

3. What are the rational numbers a and b such that if T is the temperature in degrees Fahrenheit then $aT+b$ is the temperature in degrees Celsius? (Use the internet if your general knowledge doesn't extend to this.) Show that the function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = ax + b$ for all $x \in \mathbb{R}$ is a contraction. Find the unique temperature T which is the same in both Fahrenheit and Celsius.

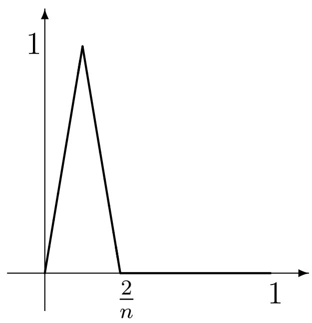
4. (Problem 6 on Chapter 4) Let θ be the identity function from $C[0, 1]$ to itself. Thus $\theta(f) = f$ for all $f \in C[0, 1]$. Show that, as a function from $(C[0, 1], d_1)$ to $(C[0, 1], d_\infty)$, θ is not continuous but that, as a function from $(C[0, 1], d_\infty)$ to $(C[0, 1], d_1)$, θ is continuous. (Use the sequential definition of continuity together with one example and one result in the notes from Chapter 2.)

Turn over

5. (Problem 2 on Chapter 5) Recall, from Problem 8 on Chapter 2, the sequence (f_n) of functions in $C[0,1]$ such that

$$f_n(x) = \begin{cases} nx \text{ if } 0 \leqslant x \leqslant \frac{1}{n}, \\ 2 - nx \text{ if } \frac{1}{n} \leqslant x \leqslant \frac{2}{n}, \\ 0 \text{ if } \frac{2}{n} \leqslant x \leqslant 1. \end{cases}$$

whose graphs are as shown in the diagram.



Let $n \geqslant 1$ and let $x = \frac{1}{n}$. Compute $f_n(x)$ and $f_{2n}(x)$. Deduce that $d_\infty(f_{2n}, f_n) \geq 1$ and hence that (f_n) is not Cauchy in $(C[0,1], d_\infty)$. Show also that if $m \geqslant 2n$ then $d_\infty(f_m, f_n) \geq 1$ and that no subsequence of (f_n) is Cauchy in $(C[0,1], d_\infty)$.

Assignment given out on Monday 27th November, for handing in on Tuesday 5th December.