

# The Positive Maximum Principle on Lie Groups

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## Abstract

We extend a classical theorem of Courrège to Lie groups in a global setting, thus characterising all linear operators on the space of smooth functions of compact support that satisfy the positive maximum principle. We show that these are Lévy type operators (with variable characteristics), and pseudo-differential operators when the group is compact. If the characteristics are constant, then the operator is the generator of the contraction semigroup associated to a convolution semigroup of sub-probability measures.

*We dedicate this paper to the memory of Herbert Heyer 1936–2018.*

## 1 Introduction

In 1965/6, Courrège [9] classified all linear operators in Euclidean space that satisfy the positive maximum principle (PMP). He first showed that these were what we now call *Lévy type operators*, in that they are the sum of a second order elliptic operator and a non-local term expressed as integration against a kernel. Secondly he showed that all such operators may be realised as pseudo-differential operators, but because of the non-local term, the symbol does not in general have the smoothness properties that are typically encountered when meeting pseudo-differential operators, say in the study of

pde's, or in index theory. Nonetheless, in recent years Courrège's theorem has found a great deal of application in the study of Feller–Markov processes, due mainly to the work of N.Jacob, R.Schilling and their collaborators (see e.g. [20, 21, 8]). The key observation is that the generator of a Feller–Markov semigroup having a sufficiently rich domain satisfies the PMP, and so will be both a Lévy type operator, and a pseudo–differential operator, whose symbol is a rich source of probabilistic information about the behavior of the process. The aim of the current paper is to extend this theory to Lie groups.

In fact, Courrège [10] has partially extended his results to get the form of generators of Feller–Markov processes on manifolds (see also [7]); but the results are given in terms of local co–ordinates, and are not presented in a form that is easy to use. Our approach is more powerful, at least in the Lie group context, in that we are able to classify all operators satisfying the PMP, and our results are global, with all differential operators being expressed in terms of a Lie algebra basis. A particular class of Feller–Markov processes of particular importance are the Lévy processes [23], and these are essentially in one–to–one correspondence with convolution semigroups of probability measures on  $G$ . The associated semigroup generator was classified by Hunt [19] in 1956. The Hunt generator, as would be expected (see [1]), has essentially the same structure as the Lévy–type operators that we obtain, but the functions and kernels that parameterise the linear operator are constant in this special case.

Of course pseudo–differential operators have been considered on manifolds since the theory was first developed; but in a local sense using local co–ordinates. In Chapter 10 of [24] Ruzhansky and Turunen have created a global theory of pseudo–differential operators on compact Lie groups, making use of Peter–Weyl theory (see also [25] for further developments). So the symbol of the operator, in that sense, is a matrix and not a scalar. This theory was applied to Hunt semigroups by one of us in [3] and some applications to Feller semigroups were developed in [4]. In the present paper we complete the programme of extending the results of [9] to (compact) Lie groups, by showing that all linear operators that satisfy the PMP may be expressed as pseudo–differential operators in the Ruzhansky–Turunen sense. As we might expect, the symbol has the form of a generalised characteristic exponent as occurs in the Lévy–Khintchine formula for convolution semigroups (see section 5.5. in [5], pp.144–6), but with variable characteristics.

We do not follow Courrège's original approach in characterising linear operators satisfying the PMP. Instead we adopt the very slick approach of Hoh [16], who reduces the problem to studying associated linear functionals that turn out to be distributions of order 2. This requires us to make use of a global theory of distributions on Lie groups, as developed by Ehrenpreis

[11]. The drawback of Hoh's approach is that it doesn't fit naturally into the framework of the Banach space comprising continuous functions that vanish at infinity, which is natural for studying Feller semigroups. However we are able to find sufficient conditions for the operator of interest to act in that context.

For Euclidean spaces, Courrège's results have led to significant progress in the study of Feller processes, not least through the solution of the martingale problem (see [16, 17] and Chapter 4 of [21]). For other results in this area see the survey [8]. Our expectation is that the work of the current paper will open up the possibility of pursuing a similar programme for processes on Lie groups.

The organisation of the paper is as follows. We first give a short account of distributions on Lie groups, which is sufficient for our purposes. Then we prove the generalised Courrège theorem. Next we examine Hunt's theorem from within this context. Finally we specialise to compact Lie groups and obtain the pseudo-differential operator representation. More detailed accounts of some of the results herein can be found in [22]. A forthcoming paper will deal with extensions of many of these results to symmetric spaces. [6]

**Notation.** Throughout this paper,  $G$  is a Lie group having neutral element  $e$ , dimension  $d$  and Lie algebra  $\mathfrak{g}$ . We denote by  $\mathcal{F}(G)$ , the space of all real-valued functions on  $G$ ,  $C_0(G)$  the Banach space (with respect to the supremum norm) of all real-valued, continuous functions on  $G$  that vanish at infinity, and  $C_c^\infty(G)$  the dense linear manifold in  $C_0(G)$  of smooth functions with compact support. The Borel  $\sigma$ -algebra of  $G$  is denoted as  $\mathcal{B}(G)$ . For each  $g \in G$ ,  $l_g$  denotes the operation of left translation by  $G$  on itself defined by  $l_g(h) = gh$ . The pullback to  $C_0(G)$  is denoted  $L_g$ , so for all  $g \in G, f \in C_0(G), L_g f = f \circ l_g$ . Note that  $l_g$  is a diffeomorphism and  $L_g$  is a Banach space (linear) automorphism.

The trace of a real or complex  $d \times d$  matrix  $A$  is written  $\text{tr}(A)$ .

## 2 Distributions on Lie Groups

We will need some facts about distributions on Lie groups. Of course Lie groups are  $C^\infty$ -manifolds, and the standard approach to defining distributions therein is to make use of local co-ordinate systems (see e.g. section 6.3 of [18]). We wish to employ the global structure on the Lie group, and in this respect we follow Ehrenpreis [11]. We do not seek to do more in this direction than develop the few tools that we will need for our purposes.

We fix once and for all a basis  $\{X_1, \dots, X_d\}$  of  $\mathfrak{g}$ . If  $S$  is a compact

subset of  $G$  we will use the notation  $\|f\|_{\infty,S} := \sup_{g \in S} |f(g)|$  for all  $f \in C_c^\infty(G)$ . Let  $\alpha = (\alpha_1, \dots, \alpha_N)$  be a multi-index where  $\alpha_i \in \mathbb{Z}_+$ , for  $i = 1, \dots, N$ , and define  $|\alpha| = \alpha_1 + \dots + \alpha_N$ . We are going to introduce the (seemingly ambiguous) notation  $X^\alpha$ , with the understanding that we will only employ it when we sum over  $\alpha$ , and that the sum will contain all distinct non-commuting combinations of products of the  $X_i$ 's (acting as differential operators on  $C^\infty(G)$ ). So for example, for all  $f \in C^\infty(G)$ ,

$$\sum_{|\alpha| \leq 2} X^\alpha f = f + X_1 f + X_2 f + X_1^2 f + X_1 X_2 f + X_2 X_1 f + X_2^2 f.$$

A *distribution*  $P$  on  $G$  is a real-valued linear functional defined on  $C_c^\infty(G)$  such that for every compact set  $K$  contained in  $G$ , there exists  $k \in \mathbb{Z}_+$ ,  $C > 0$  so that for all  $f \in C_c^\infty(K)$ ,

$$|Pf| \leq C \sum_{|\alpha| \leq k} \|X^\alpha f\|_{\infty,G}. \quad (2.1)$$

We say that  $P$  is of *order*  $k$  if the same  $k$  may be used in (2.1) for all compact  $K \subseteq G$ . We say that a distribution is *positive* if  $f \in C_c^\infty(G)$  and  $f \geq 0$  implies  $Pf \geq 0$ .

**Proposition 2.1.** *Any positive distribution on  $G$  is induced by a regular Borel measure  $\mu$  in the sense that*

$$Pf = \int_G f d\mu,$$

for all  $f \in C_c^\infty(G)$ .

*Proof.* This is proved in the same way as the corresponding result on Euclidean space. We first imitate the proof of Theorem 2.1.7 in [18] to show that any positive distribution is of order zero, and then apply the Riesz representation theorem.  $\square$

The *support*  $\text{supp}(P)$  of a distribution  $P$  is the set of all points in  $G$  which have no open neighbourhood on which the restriction of  $P$  vanishes. Let  $U$  be an (open) canonical co-ordinate neighbourhood of  $e$  with co-ordinate functions  $x_1, \dots, x_d$ , where  $x_i \in C_c^\infty(G)$  for  $i = 1, \dots, d$ . We denote by  $\phi$ , the mapping

$$g \in U \rightarrow (x_1(g), \dots, x_d(g)) \in \mathbb{R}^d.$$

Then  $\phi$  is a homeomorphism from  $U$  to an open neighbourhood  $\tilde{U}$  of  $\mathbb{R}^d$  with  $\phi(e) = 0$ . For each  $f \in C^\infty(U)$ , we write  $J_\phi f = f \circ \phi^{-1}$ . So  $J_\phi$  is a

linear isomorphism between  $C^\infty(U)$  and  $C^\infty(\tilde{U})$ . If we identify the vector field  $X_i$  with its restriction to  $U$ , we have that there exists a vector field  $\tilde{X}_i = \sum_{j=1}^d c_{ij}(\cdot)\partial_j$  on  $\tilde{U}$ , such that  $\tilde{X}_i = J_\phi X_i J_\phi^{-1}$  for all  $i = 1, \dots, d$ , where  $c_{ij} \in C^\infty(\tilde{U})$  with  $c_{ij}(0) = \delta_{ij}$ .

**Theorem 2.2.** *If  $P$  is a distribution of order  $k$  having support  $\{e\}$ , then it has the form*

$$Pf = \sum_{|\alpha| \leq k} c_\alpha X^\alpha f(e),$$

for all  $f \in C_c^\infty(G)$ , where  $c_\alpha \in \mathbb{R}$  for each multi-index  $\alpha$ .

*Proof.* If  $\text{supp}(P) = \{e\}$ , then  $P$  fails to vanish on every co-ordinate neighbourhood of  $e$ . So in particular it fails to vanish on  $U$ . If  $f \in C_c^\infty(U)$ , we write  $\tilde{f} = J_\phi f \in C_c^\infty(\tilde{U})$ . We define a linear functional  $\tilde{P} : C_c^\infty(\tilde{U}) \rightarrow \mathbb{R}$ , by  $\tilde{P}\tilde{f} = Pf$ . We then have

$$\begin{aligned} |\tilde{P}\tilde{f}| &= |Pf| \\ &\leq C \sum_{|\alpha| \leq k} \|X^\alpha f\|_{\infty, G} \\ &= C \sum_{|\alpha| \leq k} \|J_\phi X^\alpha f\|_{\infty, \mathbb{R}^d} \\ &= C \sum_{|\alpha| \leq k} \|J_\phi J_\phi^{-1} \tilde{X}^\alpha J_\phi f\|_{\infty, \mathbb{R}^d} \\ &\leq C' \sum_{|\alpha| \leq k} \|\partial^\alpha \tilde{f}\|_{\infty, \mathbb{R}^d}. \end{aligned}$$

From this we see that  $\tilde{P}$  may be extended to a distribution on  $\mathbb{R}^d$  having support  $\{0\}$ . Then by Theorem 2.3.4 in [18], we have

$$\tilde{P}\tilde{f} = \sum_{|\alpha| \leq k} d_\alpha \partial^\alpha \tilde{f}(0),$$

for  $d_\alpha \in \mathbb{R}$ ,  $|\alpha| \leq k$ . from this it follows that

$$Pf = \sum_{|\alpha| \leq k} c_\alpha X^\alpha f(e),$$

as required, with  $c_\alpha = d_\alpha/\alpha!$ , for each  $|\alpha| \leq k$ . □

### 3 The Positive Maximum Principle and a Generalised Courrège Theorem

A linear mapping  $A : C_c^\infty(G) \rightarrow \mathcal{F}(G)$  satisfies the *positive maximum principle* (PMP) if  $f \in C_c^\infty(G)$  and  $f(\rho) = \sup_{\tau \in G} f(\tau) \geq 0 \Rightarrow Af(\rho) \leq 0$ .

In this section we seek to give a global characterisation of all linear operators that satisfy the positive maximum principle (PMP). We will follow the approach developed on Euclidean space by Walter Hoh in his Habilitationsschrift [16]. First we need to extend the PMP to real-valued linear functionals  $T$  defined on the linear space  $C_c^\infty(G)$ . We say that  $T$  satisfies the *positive maximum principle* (PMP), if whenever  $f \in C_c^\infty(G)$  with  $f(e) = \sup_{\tau \in G} f(\tau) \geq 0$  then  $Tf \leq 0$ . The linear functional  $T$  is said to be *almost positive* if whenever  $f \in C_c^\infty(G)$  with  $f \geq 0$  and  $f(e) = 0$  then  $Tf \geq 0$ . It is easily verified that if  $T$  satisfies the PMP, then it is almost positive.

Next we explore the relationship between linear operators and associated linear functionals satisfying the positive maximum principle. For any linear operator  $A : C_c^\infty(G) \rightarrow \mathcal{F}(G)$ , we define a family of linear functionals  $A_g : C_c^\infty(G) \rightarrow \mathbb{R}$ ,  $g \in G$  by

$$A_g\varphi := (L_gAL_{g^{-1}})\varphi(e) = A(L_{g^{-1}}\varphi)(g). \quad (3.1)$$

**Lemma 3.1.** *A linear operator  $A : C_c^\infty(G) \rightarrow \mathcal{F}(G)$  satisfies the positive maximum principle if and only if for all  $g \in G$  the functional  $A_g$  satisfies the positive maximum principle.*

*Proof.* For necessity, let  $\varphi \in C_c^\infty(G)$  be such that  $\varphi(e) = \sup_{\sigma \in G} \varphi(\sigma) \geq 0$ . Then for any  $g \in G$ ,  $L_{g^{-1}}\varphi \in C_c^\infty(G)$  with  $L_{g^{-1}}\varphi(g) = \sup_{\sigma \in G} L_{g^{-1}}\varphi(\sigma) \geq 0$ . Now, since  $A$  satisfies the PMP we get  $A_g\varphi = AL_{g^{-1}}\varphi(g) \leq 0$ .

For sufficiency, let  $\varphi \in C_c^\infty(G)$  be such that  $\varphi(g) = \sup_{\sigma \in G} \varphi(\sigma)$  for some  $g \in G$ . Then  $L_g\varphi$  satisfies  $L_g\varphi(e) = \sup_{\sigma \in G} L_g\varphi(\sigma)$ . The functional  $A_g$  satisfies the PMP, thus we have

$$A\varphi(g) = (L_gAL_{g^{-1}}L_g)\varphi(e) = A_g(L_g\varphi) \leq 0.$$

□

The following result is an easy consequence of elementary calculus.

**Lemma 3.2.** *Any function  $\varphi \in C_c^\infty(G)$  with local maximum  $\varphi(\sigma) = \sup_{g \in V} \varphi(g)$  for some  $\sigma \in G$  on a neighbourhood  $V$  of  $\sigma$ , satisfies  $X\varphi(\sigma) = 0$  for all  $X \in \mathfrak{g}$ .*

The next result is crucial.

**Theorem 3.3.** *Let  $T : C_c^\infty(G) \rightarrow \mathbb{R}$  be an almost positive functional, then  $T$  is a distribution of order 2,*

*Proof.* Let  $\varphi \in C_c^\infty(G)$  with  $\text{supp}(\varphi) := H$ . The set  $H$  is covered by  $\bigcup l_k U$  and since it is compact, there is a finite subcover  $\{U_1, \dots, U_N\}$ , where  $U_i := l_{k_i} U$  for some  $k_i \in H$ ,  $i = 1, \dots, N$ . Furthermore, since  $G$  is a Hausdorff space, we can separate the points  $k_1, k_2, \dots, k_N$  by a family of pairwise disjoint open sets  $W_1, \dots, W_N \subset G$  such that  $k_i \in W_i$  for all  $i = 1, \dots, N$ .

Define  $\tilde{\varphi} \in C_c^\infty(G)$  by

$$\tilde{\varphi}(g) := \varphi(g) - \sum_{l=1}^N \varphi(k_l) \varepsilon_l(g) - \sum_{l=1}^N \sum_{r=1}^d x_r(k_l^{-1}g) \varepsilon_l(g) X_r \varphi(k_l) \quad (3.2)$$

for all  $g \in G$ , where for all  $l = 1, \dots, N$ ,  $\varepsilon_l \in C_c^\infty(G)$  takes values in  $[0, 1]$  with support in  $W_l$ , and is equal to 1 in a neighbourhood of  $k_l$ .

It is straightforward to check that  $\tilde{\varphi}(k_l) = X_i \tilde{\varphi}(k_m) = 0$ , for all  $l, m = 1, \dots, N$ ,  $i = 1, \dots, d$ . For any  $g \in H$ , there exists at least one  $m = 1, \dots, N$  such that  $g \in U_m$  and for such a choice of  $m$  we have

$$\tilde{\varphi}(g) = \tilde{\varphi}(g) - \tilde{\varphi}(k_m) - \sum_{i=1}^d x_i(k_m^{-1}g) X_i \tilde{\varphi}(k_m).$$

Using this with a suitably translated form of Taylor's theorem on Lie groups (see e.g. [14], p.105 or [5], pp.127–8) and the Cauchy–Schwarz inequality, we find that there exists  $C \geq 0$  so that for all  $g \in U_m$

$$\begin{aligned} |\tilde{\varphi}(g)| &= \left| \tilde{\varphi}(g) - \tilde{\varphi}(k_m) - \sum_{i=1}^d x_i(k_m^{-1}g) X_i \tilde{\varphi}(k_m) \right| \\ &\leq \frac{1}{2} \sum_{i,j=1}^d |x_i(k_m^{-1}g) x_j(k_m^{-1}g)| \sup_{v \in G} |X_i X_j \tilde{\varphi}(v)| \\ &\leq C \sum_{i=1}^d x_i(k_m^{-1}g)^2 \\ &\leq Ch(g) \end{aligned}$$

where

$$h(g) := \sum_{m=1}^N \sum_{i=1}^d x_i(k_m^{-1}g)^2 \text{ for all } g \in G.$$

The last inequality clearly extends to all  $g \in G$ . Since  $\tilde{\varphi}(e) = h(e) = 0$ , by almost positivity of  $T$  we deduce that

$$|T(\tilde{\varphi})| \leq CT(h).$$

The result then follows by the definition of  $\tilde{\varphi}$ .  $\square$

Recall that a Borel measure  $\mu$  on  $G$  is a *Lévy measure* if  $\nu(\{e\}) = 0$  and for every co-ordinate neighbourhood  $U$  of  $e$  we have

$$\int_U \left( \sum_{i=1}^d x_i^2(g) \right) \mu(dg) < \infty, \text{ and } \mu(U^c) < \infty.$$

We now establish a canonical form for linear functionals that satisfy the PMP.

**Theorem 3.4.** *Let  $T : C_c^\infty(G) \rightarrow \mathbb{R}$  be a linear functional satisfying the positive maximum principle. Then there exists  $c \geq 0$ ,  $b = (b_1, \dots, b_d) \in \mathbb{R}^d$ , a non-negative definite symmetric  $d \times d$  real-valued matrix  $a = (a_{ij})$ , and a Lévy measure  $\mu$  on  $G$  such that*

$$\begin{aligned} T\varphi &= \frac{1}{2} \sum_{i,j=1}^d a_{ij} X_i X_j \varphi(e) + \sum_{i=1}^d b_i X_i \varphi(e) - c\varphi(e) \\ &+ \int_G \left( \varphi(g) - \varphi(e) - \sum_{i=1}^d x_i(g) X_i \varphi(e) \right) \mu(dg), \end{aligned} \quad (3.3)$$

for all  $\varphi \in C_c^\infty(G)$ .

*Proof.* We begin by remarking that  $T$  is a distribution of order 2 by Theorem 3.3. Let  $V$  be a compact neighbourhood of  $e$  such that  $\bar{U} \subseteq V$ , and let  $\phi_1, \phi_2 \in C_c^\infty(G)$  be two functions taking values in  $[0, 1]$  with supports respectively in  $V$  and in  $U^c$ , such that  $\phi_1(g) = 1$  for all  $g \in U$ , and  $\phi_2(g) = 1$  for all  $g \in V^c$ . We introduce the function  $\xi := \sum_{i=1}^d x_i^2(\cdot) \cdot \phi_1 + \phi_2$  from  $G$  to  $\mathbb{R}$  and note that  $\xi(e) = 0$ . Then  $\xi \cdot T$  is a positive distribution. Indeed, for all  $\varphi \in C_c^\infty(G)$  such that  $\varphi \geq 0$ , we also have  $\xi \cdot \varphi \geq 0$  with  $\xi(e)\varphi(e) = 0$ . So by almost positivity of  $T$  we get

$$\langle \xi \cdot T, \varphi \rangle = \langle T, \xi \cdot \varphi \rangle \geq 0.$$

Since  $\xi \cdot T$  is a positive distribution, by Corollary 2.1 there exists a unique regular Borel measure  $\nu$  on  $G$  such that  $\nu = \xi \cdot T$ , i.e.  $\langle \xi \cdot T, f \rangle = \int_G f(g) \nu(dg)$



for all  $f \in C_c^\infty(G)$ . We define another Borel measure  $\mu$  by  $\mu(\{e\}) = 0$  and  $\mu := \frac{1}{\xi} \cdot \nu|_{G^*}$  on  $G^* := G \setminus \{e\}$ , and we have

$$\langle T, f \rangle = \langle \xi \cdot T, \frac{1}{\xi} f \rangle = \int_G f(g) \frac{1}{\xi(g)} \nu(dg) = \int_G f(g) \mu(dg), \quad \text{for all } f \in C_c^\infty(G). \quad (3.4)$$

We show that  $\mu$  is a Lévy measure. By regularity of  $\nu$  we have

$$\int_U \left( \sum_{i=1}^d x_i^2(g) \right) \mu(dg) \leq \int_V \xi(g) \mu(dg) = \nu(V) < \infty \quad (3.5)$$

Next let  $\alpha, \beta \in C_c^\infty(G)$  be two functions taking values in  $[0, 1]$ , such that  $\text{supp}(\alpha) \subset U$  and  $\text{supp}(\beta) \subset U^c$  with  $\alpha(e) = 1$ , so

$$\alpha(e) + \beta(e) = 1 + 0 = \sup_{g \in G} (\alpha + \beta)(g).$$

By the positive maximum principle,  $T(\alpha + \beta) \leq 0$ , so by linearity of  $T$ , we get  $T\beta \leq -T\alpha$ , that is

$$\langle T, \beta \rangle = \int_{G^*} \beta(g) \mu(dg) \leq -\langle T, \alpha \rangle$$

Taking the supremum over all possible  $\beta$ , it then follows that

$$\mu(U^c) \leq -\langle T, \alpha \rangle < \infty \quad (3.6)$$

So the measure  $\mu$  is indeed a Lévy measure.

Our next goal is to derive the form (3.3). To this end we introduce a linear functional  $S : C_c^\infty(G) \rightarrow \mathbb{R}$ , by

$$S\varphi := \int_{G^*} \left[ \varphi(g) - \varphi(e) - \sum_{i=1}^d x_i(g) X_i \varphi(e) \right] \mu(dg), \quad \text{for all } \varphi \in C_c^\infty(G).$$

The integral is well defined since all functions in the integrand have compact support. By Lemma 3.2 for all  $\varphi \in C_c^\infty(G)$  such that  $\varphi(e) = \sup_{g \in G} \varphi(g) \geq 0$

we have  $X_i \varphi(e) = 0$ , therefore

$$S\varphi = \int_{G^*} (\varphi(g) - \varphi(e)) \mu(dg) \leq 0.$$

That is,  $S$  satisfies the positive maximum principle. By Theorem 3.3 both  $T$  and  $S$  are distributions of order 2, thus so is their difference  $P := T - S$ . If  $\varphi \in C_c^\infty(G)$  with  $e \notin \text{supp}(\varphi)$ , then

$$S\varphi = \int_{G^*} \varphi(g) \mu(dg) = T\varphi.$$

That is  $P\varphi = 0$  for all  $\varphi \in C_c^\infty(G)$ , therefore  $\text{supp}(P) \subseteq \{e\}$ . Thus, using Theorem 2.2,  $P$  is of the form

$$P\varphi = \frac{1}{2} \sum_{i,j=1}^d a_{ij} X_i X_j \varphi(e) + \sum_{i=1}^d b_i X_i \varphi(e) - c\varphi(e).$$

We will now prove that the constant  $c$  is positive. Let  $(\varphi_k)_{k \in \mathbb{N}}$  be a sequence of non-negative, monotone increasing functions in  $C_c^\infty(G)$  such that  $\varphi_k = 1$  in a neighbourhood of  $e$  which are pointwise convergent to  $\mathbb{1}_G$ , then by the monotone convergence theorem

$$S\varphi_k = \int_{G^*} (\varphi_k(g) - 1) \mu(dg) \rightarrow 0, \quad \text{as } k \rightarrow \infty.$$

So

$$T\varphi_k = P\varphi_k + S\varphi_k = -c\varphi_k(e) + S\varphi_k \rightarrow -c, \quad \text{as } k \rightarrow \infty.$$

Note that for all  $k \in \mathbb{N}$ ,  $\varphi_k(e) = \sup_{g \in G} \varphi_k(g) \geq 0$ , and since  $T$  satisfies the positive maximum principle we have  $T\varphi_k \leq 0$ . Thus,  $c \geq 0$ .

It is clear that  $(a_{ij})$  is symmetric. We will now prove that  $(a_{ij})$  is also positive definite. Let  $(\varepsilon_k)_{k \in \mathbb{N}}$  be a sequence of monotone decreasing functions in  $C_c^\infty(G)$ , taking values in  $[0, 1]$  such that  $\varepsilon_k = 1$  in a neighbourhood  $e$  with  $V_k \subset V_{k'}$  when  $k > k'$  and  $\bigcap_{k \in \mathbb{N}} V_k = \{e\}$ . Let us also denote by  $f_\xi \in C_c^\infty(G)$  the

function defined by  $f_\xi(\cdot) := \frac{1}{2} \sum_{i,j=1}^d \xi_i \xi_j x_i(\cdot) x_j(\cdot)$ , where  $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$ .

Note that for all  $1 \leq k, l \leq d$ , we have

$$\begin{aligned} X_l X_k f_\xi &= \frac{1}{2} \sum_{i,j=1}^d \xi_i \xi_j X_l X_k (x_i x_j) \\ &= \frac{1}{2} \sum_{i,j=1}^d \xi_i \xi_j (\delta_{ki} \delta_{lj} + \delta_{kj} \delta_{li}) \\ &= \frac{1}{2} (\xi_k \xi_l + \xi_l \xi_k) = \xi_l \xi_k. \end{aligned}$$

And we have

$$\begin{aligned} T(\varepsilon_k \cdot f_\xi) &= P(\varepsilon_k \cdot f_\xi) + S(\varepsilon_k \cdot f_\xi) \\ &= \frac{1}{2} \sum_{i,j=1}^d a_{ij} \xi_i \xi_j + \int_{G^*} \varepsilon_k(g) f_\xi(g) \mu(dg). \end{aligned}$$

Thus, by dominated convergence  $T(\varepsilon_k \cdot f) \rightarrow \frac{1}{2} \sum_{i,j=1}^d a_{ij} \xi_i \xi_j$  as  $k$  goes to infinity.

Also, since  $T$  is almost positive,  $T(\varepsilon_k \cdot f) \geq 0$ , so  $\sum_{i,j=1}^d a_{ij} \xi_i \xi_j \geq 0$  and  $(a_{ij})$  is positive definite.

Furthermore, it is clear that  $(a_{ij})$  and  $c$  are uniquely defined for all  $T$ . Moreover,  $\nu$  was uniquely defined and therefore  $\mu$  is also uniquely defined. This allows us to calculate  $\sum_{i=1}^d b_i X_i \varphi(e)$  for any  $\varphi \in C_c^\infty(G)$ , so the vector  $b = (b_1, \dots, b_d) \in \mathbb{R}^d$  is also uniquely defined.  $\square$

Next we establish the converse to Theorem 3.4.

**Theorem 3.5.** *Every linear functional on  $C_c^\infty(G)$  of the form (3.3) satisfies the positive maximum principle.*

*Proof.* By Lemma 3.2, for any function  $\varphi \in C_c^\infty(G)$  such that  $\varphi(e) = \sup_{g \in G} \varphi(g) \geq 0$ , we have  $X\varphi(e) = 0$  for all  $X \in \mathfrak{g}$ . Thus, in (3.3) both the first order differential part and the integral part satisfy the positive maximum principle. So now we only have to deal with the second order differential part.

Fix  $h \in U$  and let us consider the function  $H : t \mapsto \varphi \left( \exp \left( t \sum_{i=1}^d x_i(h) X_i \right) \right)$  from  $\mathbb{R}$  to  $G$ . Let  $P_2$  be the second order Taylor polynomial of  $H$  around 0,

$$P_2(t) = \varphi(e) + t \sum_{i=1}^d x_i(h) X_i \varphi(e) + \frac{t^2}{2} \sum_{i,j=1}^d x_i(h) x_j(h) X_i X_j \varphi(e)$$

Then  $H(t) - P_2(t) = o(t^2)$  as  $t \rightarrow 0$ . That is

$$\varphi \left( \exp \left( t \sum_{i=1}^d x_i(h) X_i \right) \right) - \varphi(e) - t \sum_{i=1}^d x_i(h) X_i \varphi(e) - \frac{t^2}{2} \sum_{i,j=1}^d x_i(h) x_j(h) X_i X_j \varphi(e) = o(t^2)$$

Thus, by Lemma 3.2 and the fact that  $\varphi(e) = \sup_{g \in G} \varphi(g)$  we have

$$o(t^2) + \frac{t^2}{2} \sum_{i,j=1}^d x_i(h) x_j(h) X_i X_j \varphi(e) = \varphi \left( \exp \left( t \sum_{i=1}^d x_i(h) X_i \right) \right) - \varphi(e) \leq 0$$

Hence, dividing by  $t^2$  and taking the limit  $t \rightarrow 0$  we get

$$\sum_{i,j=1}^d x_i(h)x_j(h)X_iX_j\varphi(e) \leq 0$$

The map  $h \mapsto (x_1(h), \dots, x_d(h))$  is a diffeomorphism from  $U$  to an open neighbourhood  $\tilde{U}$  of  $0 \in \mathbb{R}^d$ . Let us fix an open ball  $B_r(0) \subset \tilde{U}$  of radius  $r > 0$ . Then given any  $\lambda \in \mathbb{R}^d$ , there exists  $c(\lambda) > 0$  such that  $c(\lambda)\lambda \in B_r(0)$ , i.e. for all  $i = 1, \dots, d$ ,  $c(\lambda)\lambda_i = x_i(h)$  for some  $h \in U$ . Then we have

$$\sum_{i,j=1}^d \lambda_i\lambda_jX_iX_jf(e) = \frac{1}{c(\lambda)^2} \sum_{i,j=1}^d x_i(h)x_j(h)X_iX_jf(e) \leq 0$$

We conclude that any linear functional on  $C_c^\infty(G)$  of the form (3.3) satisfies the positive maximum principle.  $\square$

We may now proceed to the main result of this section, namely the generalisation of Courrège's theorem from Euclidean space to Lie groups. We will need the concept of a *Lévy kernel* on  $G$  which is a mapping  $\mu : G \times \mathcal{B}(G) \rightarrow [0, \infty]$  such that  $\mu(g, \cdot)$  is a Lévy measure on  $G$  for all  $g \in G$ .

**Theorem 3.6.** *The mapping  $A : C_c^\infty(G) \rightarrow \mathcal{F}(G)$  satisfies the PMP if and only if there exist functions  $c, b_i, a_{jk}$  ( $1 \leq i, j, k \leq d$ ) from  $G$  to  $\mathbb{R}$ , wherein  $c$  is non-negative, and the matrix  $a(\sigma) := (a_{jk}(\sigma))$  is non-negative definite and symmetric for all  $\sigma \in G$ , and a Lévy kernel  $\mu$ , such that for all  $f \in C_c^\infty(G), \sigma \in G$ ,*

$$\begin{aligned} Af(\sigma) &= -c(\sigma)f(\sigma) + \sum_{i=1}^d b_i(\sigma)X_i f(\sigma) + \sum_{j,k=1}^d a_{jk}(\sigma)X_jX_k f(\sigma) \\ &+ \int_G \left( f(\sigma\tau) - f(\sigma) - \sum_{i=1}^d x_i(\tau)X_i f(\sigma) \right) \mu(\sigma, d\tau). \end{aligned} \quad (3.7)$$

*Proof.* From Lemma 3.1, we know that  $A$  satisfies the PMP if and only if for all  $g \in G$  the linear functional  $A_g$  as defined in (3.1) satisfies the PMP. Thus, for each  $g \in G$ , the functional  $A_g$  has the form (3.3),

$$\begin{aligned} A_g\varphi &= \frac{1}{2} \sum_{i,j=1}^d a_{ij}(g)X_iX_j\varphi(e) - \sum_{i=1}^d b_i(g)X_i\varphi(e) - c(g)\varphi(e) \\ &+ \int_G \left( \varphi(\tau) - \varphi(e) \sum_{i,j=1}^d x_i(\tau)X_i\varphi(e) \right) \mu(g, d\tau). \end{aligned}$$

Then in particular for the function  $L_g\varphi \in C_c^\infty(G)$ , we get

$$\begin{aligned} A\varphi(g) = A_g(L_g\varphi) &= \frac{1}{2} \sum_{i,j=1}^d a_{ij}(g) X_i X_j \varphi(g) - \sum_{i=1}^d b_i(g) X_i \varphi(g) - c(g) \varphi(g) \\ &\quad + \int_G \left( \varphi(g\tau) - \varphi(g) - \sum_{i=1}^d x_i(\tau) X_i \varphi(g) \right) \mu(g, d\tau), \end{aligned}$$

and the result is established.  $\square$

In the last part of this section, we will establish sufficient conditions for  $A : C_c^\infty(G) \rightarrow C_0(G)$ . We introduce some simplifying notation. Define  $H : C_c^\infty(G) \rightarrow C(G \times G)$  by

$$Hf(g, \tau) := f(g\tau) - f(g) - \sum_{i=1}^d x_i(\tau) X_i f(g), \quad (3.8)$$

for all  $f \in C_c^\infty(G)$ ,  $g, \tau \in G$ . We then write  $A = A_1 + A_2 + A_3$  in (3.7), where  $A_2 f(g) = \int_U Hf(g, \tau) \mu(g, d\tau)$ , and  $A_3 f(g) = \int_{U^c} Hf(g, \tau) \mu(g, d\tau)$ ,

**Theorem 3.7.** *If  $A : C_c^\infty(G) \rightarrow \mathcal{F}(G)$  satisfies the positive maximum principle such that in (3.7)*

i) *The functions  $a_{ij}, b_k, c$  are continuous for all  $i, j, k = 1, \dots, d$*

ii) *If  $(g_n)_{n \in \mathbb{N}}$  is a sequence such that  $g_n \rightarrow g$  as  $n \rightarrow \infty$  for some  $g \in G$ , then<sup>1</sup>*

$$\lim_{n \rightarrow \infty} \int_U \sum_{i=1}^d x_i^2(\tau) |\mu(g, d\tau) - \mu(g_n, d\tau)| = 0$$

and

$$\lim_{n \rightarrow \infty} \int_{U^c} |\mu(g, d\tau) - \mu(g_n, d\tau)| = 0,$$

then the  $\text{Ran}(A) \subseteq C(G)$ .

*Proof.* It is clear that if (i) holds, the differential operator term  $A_1$  in (3.7) has continuous range, and so we concentrate on  $A_3$  and  $A_2$  (in that order). Let  $(g_n)$  be a sequence in  $G$  converging to  $g \in G$  as  $n \rightarrow \infty$ . For each  $n \in \mathbb{N}$ , we have

$$\begin{aligned} &|A_3 f(g) - A_3 f(g_n)| \\ &\leq \left| \int_{U^c} \left[ f(g\tau) - f(g_n\tau) - f(g) + f(g_n) - \sum_{i=1}^d x_i(\tau) (X_i f(g) - X_i f(g_n)) \right] \mu(g, d\tau) \right| \\ &+ \left| \int_{U^c} Hf(g_n, \tau) [\mu(g_n, d\tau) - \mu(g, d\tau)] \right| \end{aligned}$$

<sup>1</sup>Here we have abused notation somewhat, in that for each  $n \in \mathbb{N}$ , we should understand the integral as being with respect to the variation of the signed measure  $\mu(g, \cdot) - \mu(g_n, \cdot)$ .

Hence, as  $n \rightarrow \infty$ , the first term on the right hand side of the display converges to 0 by dominated convergence and assumption (ii). For the second term, we have an upper bound

$$\left| \int_{U^c} Hf(g_n, \tau) [\mu(g_n, d\tau) - \mu(g, d\tau)] \right| \leq K_f \int_{U^c} |\mu(g_n, d\tau) - \mu(g, d\tau)|$$

where  $K_f := 2 \sup_{g \in G} |f(g)| + \sum_{i=1}^d \sup_{g \in G} |x_i(g)| |X_i f(g)|$ . Using (ii), the right hand side converges to 0 as  $n \rightarrow \infty$ .

We will now deal with the operator  $A_2$ . For all  $g \in G$ , we have

$$\begin{aligned} & |A_2 f(g) - A_2 f(g_n)| \\ & \leq \left| \int_U [Hf(g, \tau) - Hf(g_n, \tau)] \mu(g, d\tau) \right| + \left| \int_U Hf(g_n, \tau) [\mu(g_n, d\tau) - \mu(g, d\tau)] \right| \end{aligned}$$

By continuity, as  $n \rightarrow \infty$  we have  $Hf(g_n, \tau) \rightarrow Hf(g, \tau)$ , for all  $\tau \in U$ . Also, by Taylor's formula on Lie groups, for all  $\tau \in U$  there exist  $\tau', \tau'' \in U$  such that

$$|Hf(g, \tau) - Hf(g_n, \tau)| = \left| \frac{1}{2} \sum_{i,j=1}^d x_i(\tau) x_j(\tau) [X_i X_j f(g\tau') - X_i X_j f(g_n\tau'')] \right|$$

Let us denote  $M := \max_{i,j=1,\dots,d} \sup_{g \in G} |X_i X_j f(g)|$ . Then from the previous equation, and by the Cauchy-Schwarz inequality, we get

$$|Hf(g, \tau) - Hf(g_n, \tau)| \leq Md \sum_{i=1}^d x_i^2(\tau)$$

The right hand side is integrable over  $U$  with respect to the measure  $\mu(g, \cdot)$ , therefore by dominated convergence  $\int_U [Hf(g, \tau) - Hf(g_n, \tau)] \mu(g, d\tau)$  goes to 0 as  $n \rightarrow \infty$ . For the last integral in the display, just as before, we use Taylor's formula on Lie groups and the Cauchy-Schwarz inequality to obtain

$$|Hf(g_n, \tau)| \leq \frac{d}{2} M \sum_{i=1}^d x_i^2(\tau), \quad \text{for all } g_n \in G. \quad (3.9)$$

Then use of assumption (ii) enables us to conclude that  $A_2 f$  is continuous, and so  $Af$  is continuous on  $G$ , as required.  $\square$

**Theorem 3.8.** *If  $A : C_c^\infty(G) \rightarrow \mathcal{F}(G)$  satisfies the positive maximum principle such that in (3.7) we have*

*iii)*

$$\lim_{\sigma \rightarrow \infty} \int_U \left( \sum_{i=1}^d x_i^2(\tau) \right) \mu(\sigma, d\tau) = 0$$

*and*

$$\lim_{\sigma \rightarrow \infty} \mu(\sigma, U^c) = 0,$$

*then  $Af$  vanishes at infinity for all  $f \in C_c^\infty(G)$ .*

*Proof.* Since differential operators cannot increase supports, it is sufficient to consider the operator  $A_2 + A_3$ . Using the inequality (3.9), we obtain for all  $\sigma \in G$ ,

$$|A_2 f(\sigma)| \leq \int_U |Hf(\sigma, \tau)| \mu(\sigma, d\tau) \leq \frac{d}{2} M \int_U \left( \sum_{i=1}^d x_i^2(\tau) \right) \mu(\sigma, d\tau)$$

Therefore, by the condition (iii), given any  $\varepsilon > 0$ , there is a compact set  $B_1$  such that for all  $\sigma \in B_1^c$ ,

$$|A_2 f(\sigma)| < \varepsilon/2.$$

Using the notation from the proof of Theorem 3.7 we have

$$|A_3 f(\sigma)| \leq K_f \cdot \mu(\sigma, U^c).$$

Using condition (iii), there is a compact set  $B_2$  such that for all  $\sigma \in B_2^c$ ,

$$|A_3 f(\sigma)| < \varepsilon/2$$

Since  $B_1 \cup B_2$  is compact, we get

$$|A_2 f(\sigma) + A_3 f(\sigma)| < \varepsilon \quad \text{for all } \sigma \in (B_1 \cup B_2)^c,$$

and the proof is complete. □

Combining the results of the last two theorems, we obtain

**Corollary 3.9.** *If  $A : C_c^\infty(G) \rightarrow \mathcal{F}(G)$  satisfies the positive maximum principle and conditions (i) and (ii) from Theorem 3.7 and condition (iii) from Theorem 3.8 are satisfied, then  $A$  maps  $C_c^\infty(G)$  to  $C_0(G)$ .*

## 4 The Generalised Courrège Theorem and Killed Hunt's formula

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space where the  $\sigma$ -algebra  $\mathcal{F}$  is equipped with a filtration  $(\mathcal{F}_t)_{t \geq 0}$ . We will consider a Markov process  $(Y_t)_{t \geq 0}$  with respect to the filtration  $(\mathcal{F}_t)_{t \geq 0}$ , taking values in  $G$ . We will denote the transition probabilities of  $(Y_t)_{t \geq 0}$  by  $p_t(\sigma, A) := P(Y_t \in A | Y_0 = \sigma)$  for all  $A \in \mathcal{B}(G)$  and  $\sigma \in G$ . We then have a one-parameter contraction semigroup of operators on  $B_b(G)$  given for each  $f \in B_b(G), \sigma \in G$  by

$$T_t f(\sigma) = \int_G f(\tau) p_t(\sigma, d\tau). \quad (4.1)$$

The family of operators  $(T_t)_{t \geq 0}$  is called a *Feller semigroup* if

- i)  $T_t C_0(G) \subseteq C_0(G)$  for all  $t \geq 0$ ,
- ii)  $\lim_{t \rightarrow 0} \|T_t f - f\|_\infty = 0$  for all  $f \in C_0(G)$ .

In this case we say that  $(Y_t)_{t \geq 0}$  is a Feller process. We will denote by  $A$  the infinitesimal generator of the Feller semigroup  $(T_t)_{t \geq 0}$ . The following lemma is well-known, see [20], p.332. We will include the proof for completeness.

**Lemma 4.1.** *If  $(T_t)_{t \geq 0}$  is a Feller semigroup with generator  $A$  such that  $C_c^\infty(G) \subseteq \text{Dom}(A)$ , then  $A$  satisfies the positive maximum principle.*

*Proof.* Let  $f \in C_c^\infty(G)$  be such that  $f(g_0) = \sup_{\sigma \in G} f(\sigma) \geq 0$  for some  $g_0 \in G$ .

We have

$$Af(g_0) = \lim_{t \rightarrow 0} \frac{(T_t f - f)(g_0)}{t} = \lim_{t \rightarrow 0} \frac{1}{t} \int_G [f(\tau) - f(g_0)] p_t(g_0, d\tau) \leq 0.$$

This proves that  $A$  satisfies the PMP. □

A finite Borel measure on  $G$  is a sub-probability measure if its total mass does not exceed one, and the family  $(\mu_t)_{t \geq 0}$  of sub-probability measures on  $(G, \mathcal{B}(G))$ , is called a *convolution semigroup of sub-probability measures* if

- i)  $\mu_{s+t} = \mu_s * \mu_t$ , for all  $s, t \geq 0$
- ii)  $\mu_0 = \delta_e$ ,
- iii)  $\lim_{t \rightarrow 0} \mu_t = \delta_e$ , in the sense of weak convergence,



where  $*$  denotes the usual convolution of measures (see e.g. section 4.1 of [5] pp.82–3). We have a convolution semigroup of probability measures if  $\mu_t(G) = 1$  for all  $t \geq 0$ . The latter arise as the laws of Lévy processes on  $G$ , i.e. processes with stationary and independent increments that are stochastically continuous, see e.g. p.10 in [23]. A Lévy process is a Feller process and the contraction semigroup,  $(T_t)_{t \geq 0}$  defined on  $C_0(G)$ , is called a *Hunt semigroup* and is defined for all  $t \geq 0, f \in C_0(G), \sigma \in G$  by

$$T_t f(\sigma) = \int_G f(\sigma\tau) \mu_t(d\tau), \quad (4.2)$$

If we compare this to (4.1), we see that

$$p_t(\sigma, A) = \mu_t(\sigma^{-1}A)$$

for all  $\sigma \in G$  and  $A \in \mathcal{B}(G)$ . The infinitesimal generator  $\mathcal{L}$  of a Hunt semigroup is called the *Hunt generator*. From the definition of the Hunt semigroup it follows that  $L_\sigma T_t = T_t L_\sigma$  for all  $\sigma \in G$  and  $t \geq 0$ . Therefore, since by Lemma 5.3.2 in [5], for all  $f \in \text{Dom}(\mathcal{L}), \sigma \in G, L_\sigma f \in \text{Dom}(\mathcal{L})$  then  $L_\sigma \mathcal{L}f = \mathcal{L}L_\sigma f$ .

The Hunt semigroups are precisely the (left) translation invariant Feller semigroups, as the next result shows.

**Proposition 4.2.** *If  $(T_t, t \geq 0)$  is a Feller semigroup in  $C_0(G)$  such that  $p_0(e, \cdot) = \delta_e(\cdot)$  and  $L_\sigma T_t = T_t L_\sigma$  for all  $t \geq 0, \sigma \in G$ , then  $(T_t, t \geq 0)$  is the Hunt semigroup associated with a convolution semigroup of probability measures.*

This is proved exactly as in the Euclidean case (see e.g. Theorem 3.3.1 in [2] pp.161–2).

We have the following celebrated classification of the Hunt generator, originally due to Hunt [19]. First we define the space

$$C_0^{(2)}(G) := \{f \in C_0(G); X_i f \in C_0(G) \text{ and } X_j X_k f \in C_0(G) \text{ for all } 1 \leq i, j, k \leq d\}.$$

Clearly  $C_c^\infty(G) \subseteq C_0^{(2)}(G)$ , and so  $C_0^{(2)}(G)$  is dense in  $C_0(G)$ .

**Theorem 4.3** (Hunt's theorem). *If  $(\mu_t, t \geq 0)$  is a convolution semigroup of probability measures in  $G$  with generator  $\mathcal{L}$ , then*

1.  $C_0^{(2)}(G) \subseteq \text{Dom}(\mathcal{L})$ .

2. For each  $\sigma \in G, f \in C_0^{(2)}(G)$ ,

$$\begin{aligned} \mathcal{L}f(\sigma) &= \sum_{i=1}^d b_i X_i f(\sigma) + \frac{1}{2} \sum_{i,j=1}^d a_{ij} X_i X_j f(\sigma) \\ &+ \int_G \left( f(\sigma\tau) - f(\sigma) - \sum_{i=1}^d x_i(\tau) X_i f(\sigma) \right) \nu(d\tau), \end{aligned} \quad (4.3)$$

where  $b = (b_1, \dots, b_d) \in \mathbb{R}^d, a = (a_{ij})$  is a non-negative definite, symmetric  $d \times d$  real-valued matrix and  $\nu$  is a Lévy measure on  $G$ .

Conversely, any linear operator with a representation as in (4.3) is the restriction to  $C_0^{(2)}(G)$  of the Hunt generator corresponding to a unique convolution semigroup of probability measures.

For a proof of this result see e.g. [19, 15, 23].

Let  $c > 0$  and  $(\mu_t, t \geq 0)$  be a convolution semigroup of probability measures in  $G$ , and consider the family of measures  $(\tilde{\mu}_t, t \geq 0)$  where for each  $t \geq 0, \tilde{\mu}_t = e^{-ct} \mu_t$ . Then  $(\tilde{\mu}_t, t \geq 0)$  is a convolution semigroup of sub-probability measures, since for each  $t > 0, \tilde{\mu}_t(G) = e^{-ct} < 1$ . We obtain a  $C_0$ -contraction semigroup  $(S_t, t \geq 0)$  on  $C_0(G)$  given by  $S_t = e^{-ct} T_t$  for each  $t \geq 0$ , and we have  $S_t f(\sigma) = \int_G f(\sigma\tau) \tilde{\mu}_t(d\tau)$  for all  $f \in C_0(G), \sigma \in G$ . Clearly the action of the infinitesimal generator  $\mathcal{M}$  of this semigroup is given by  $\mathcal{M}f = -cf + \mathcal{L}f$ , for all  $f \in C_0^{(2)}(G)$ . Note that  $(S_t, t \geq 0)$  is not a Feller semigroup in our sense. It can be naturally associated with a *killed Lévy process* whose state space is the one-point compactification of  $G$ , and  $c$  is then interpreted as a killing rate. For details see e.g. [2] pp.405–6. We thus call  $\mathcal{M}$  a *killed Hunt generator*. It is clear, e.g. by the argument of the proof of Lemma 4.1 that it satisfies the PMP. In the converse direction, we have the following result.

**Theorem 4.4.** *If  $A : C_c^\infty(G) \rightarrow C(G)$  satisfies the positive maximum principle and is such that  $L_g A f = A L_g f$  for all  $f \in C_c^\infty(G), g \in G$  then*

$$\begin{aligned} A f(g) &= \frac{1}{2} \sum_{i,j=1}^d a_{ij} X_i X_j f(g) + \sum_{i=1}^d b_i X_i f(g) - c f(g) \\ &+ \int_G \left( f(g\tau) - f(g) - \sum_{i=1}^d x_i(\tau) X_i f(g) \right) \mu(d\tau), \end{aligned} \quad (4.4)$$

where  $\{a_{ij}\}$  is a non-negative definite, symmetric matrix,  $(b_1, \dots, b_d) \in \mathbb{R}^d$ ,  $c \geq 0$  and  $\mu$  is a Lévy measure on  $(G, \mathcal{B}(G))$ . Furthermore  $A$  extends to the killed Hunt generator associated to a unique convolution semigroup of sub-probability measures on  $G$ .

*Proof.* Since  $A$  satisfies the PMP, by Theorem 3.6 it is of the form (3.7). Furthermore,  $A$  is invariant under left translation on  $C_c^\infty(G)$ . Thus,

$$\begin{aligned} Af(g) &= AL_g f(e) = L_g Af(e) \\ &= \frac{1}{2} \sum_{i,j=1}^d a_{ij}(e) X_i X_j f(g) + \sum_{i=1}^d b_i(e) X_i f(g) - c(e) f(g) \\ &\quad + \int_G \left[ f(g\tau) - f(g) - \sum_{i=1}^d x_i(\tau) X_i f(g) \right] \mu(e, d\tau). \end{aligned}$$

The form (4.4) follows when we define  $a_{ij} = a_{ij}(e)$ ,  $b_k = b_k(e)$ ,  $c = c(e)$  for all  $i, j, k = 1, \dots, d$ , and  $\mu(\cdot) = \mu(e, \cdot)$ . Next write (4.4) as  $Af = -cf + Bf$ . It is clear that  $A$  and  $B$  may be extended to linear operators  $A_1$  and  $B_1$  on  $C_0^{(2)}(G)$ , so that  $A_1 f = -cf + B_1 f$  for all  $f \in C_0^{(2)}(G)$ . Then by Hunt's theorem,  $B_1$  extends to the Hunt generator associated to a unique convolution semigroup  $(\mu_t, t \geq 0)$  of probability measures, and then the required convolution semigroup of sub-probability measures is given, as above, by defining  $\tilde{\mu}_t = e^{-ct} \mu_t$  for each  $t \geq 0$ . Moreover by Theorem 5.3.4 on pp.137 of [2],  $C_c^\infty(G)$  is a core for the Hunt generator, and so for the killed Hunt generator, from which we see that the action of  $A$  on  $C_c^\infty(G)$  uniquely determines  $(\tilde{\mu}_t, t \geq 0)$ .  $\square$

## 5 Pseudo-Differential Operator Representation

In this section, we assume that the conditions of Corollary 3.9 hold so that  $A$  maps  $C_c^\infty(G)$  to  $C_0(G)$ . Since any operator in  $C_0(G)$  that satisfies the positive maximum principle is closeable (see e.g. [12] Lemma 2.11, p.16), there is no harm done in assuming that  $A$  is closed (on its maximal domain) and we make this assumption henceforth.

We will also assume that  $G$  is compact. It will be equipped with normalised bi-invariant Haar measure. Let  $\widehat{G}$  be the *unitary dual* of  $G$ , i.e. the set of all (equivalence classes with respect to unitary conjugation) of irreducible unitary representations of  $G$ . If  $\pi \in \widehat{G}$  then  $\pi$  acts as unitary matrices on a complex Hilbert space  $V_\pi$  having dimension  $d_\pi < \infty$ . We denote

the associated *derived representation* of  $\mathfrak{g}$ , acting as skew-Hermitian matrices on  $V_\pi$  by  $d\pi$ . Choosing bases, once and for all, in  $V_\pi$ , for each  $\pi \in G$ , we may define the co-ordinate functions in the usual way as  $\pi_{ij} : G \rightarrow \mathbb{C}$ , given by  $\pi_{ij}(g) := \pi(g)_{ij}$  for each  $1 \leq i, j \leq d_\pi$ . These mappings are known to be  $C^\infty$ . Taking  $f = \pi_{ij}$  in (3.7), we may for each  $\sigma \in G$ , define the matrix  $A\pi(\sigma) = A\pi_{ij}(\sigma)$ . Define the matrix valued function

$$j_A(\sigma, \pi) := \pi(\sigma)^{-1} A\pi(\sigma).$$

**Proposition 5.1.** *For each  $\pi \in \widehat{G}, \sigma \in G$ ,*

$$\begin{aligned} j_A(\sigma, \pi) = & -c(\sigma)I_\pi + \sum_{i=1}^d b_i(\sigma)d\pi(X_i) + \frac{1}{2} \sum_{j,k=1}^d a_{jk}(\sigma)d\pi(X_j)d\pi(X_k) \\ & + \int_G \left( \pi(\tau) - I_\pi - \sum_{i=1}^d x_i(\tau)d\pi(X_i) \right) \nu(\sigma, d\tau). \end{aligned} \quad (5.1)$$

*Proof.* This follows in a straightforward manner from (3.7) and the definition of  $j$ , using standard properties of representations, and the fact that (with an obvious notation),

$$X\pi(\sigma) = \left. \frac{d}{du} \pi(\sigma \exp(uX)) \right|_{u=0} = \pi(\sigma)d\pi(X),$$

for all  $X \in \mathfrak{g}$ . □

We want to show that  $A$  is a pseudo-differential operator with symbol  $j_A$ . Define  $\mathcal{E}(G)$  to be the linear span of  $S(G) := \{\sqrt{d_\pi}\pi_{ij}; 1 \leq i, j \leq d_\pi, \pi \in \widehat{G}\}$ . Then by the Peter-Weyl theorem,  $S(G)$  is a complete orthonormal basis for  $L^2(G)$ , and  $\mathcal{E}(G)$  is norm dense in  $L^2(G)$ , and uniformly dense in  $C(G)$ . For all  $f \in \mathcal{E}(G), \sigma \in G$  we have the Fourier expansion

$$f(\sigma) = \sum_{\pi \in \widehat{G}} d_\pi \text{tr}(\widehat{f}(\pi)\pi(\sigma)), \quad (5.2)$$

and the right hand side of (5.2) is in fact a finite sum. The matrix  $\widehat{f}(\pi) = \int_G f(\tau^{-1})\pi(\tau)d\tau$  is the Fourier transform of  $f$ .

Using the definition of  $j_A$ , we obtain

$$\begin{aligned}
Af(\sigma) &= \sum_{\pi \in \widehat{G}} d_\pi \operatorname{tr}(\widehat{f}(\pi) A \pi(\sigma)) \\
&= \sum_{\pi \in \widehat{G}} d_\pi \operatorname{tr}(\widehat{f}(\pi) \pi(\sigma) j_A(\sigma, \pi)) \\
&= \sum_{\pi \in \widehat{G}} d_\pi \operatorname{tr}(j_A(\sigma, \pi) \widehat{f}(\pi) \pi(\sigma)). \tag{5.3}
\end{aligned}$$

We need to show that the right hand side of (5.3) converges to  $Af(\sigma)$  for all  $f \in C^\infty(G)$ . In order to investigate this question, we will need the space  $D$  of all dominant weights on  $G$  and we define  $D_0 := D \setminus \{0\}$ . Recall that these are in one-to-one correspondence with elements of  $\widehat{G}$ , so that each  $\lambda \in D$  is mapped to a unique  $\pi_\lambda \in \widehat{G}$ . We will equip  $\mathfrak{g}$  with an Ad-invariant inner product, and write the associated norm as  $|\cdot|$ . This induces a norm on  $D$  which is denoted by the same symbol. All results that follow are taken from [26] (see also Chapter 3 of [5]). Writing  $d_\lambda := d_{\pi_\lambda}$ , we have the useful estimates

$$d_\lambda \leq C_1 |\lambda|^m, \tag{5.4}$$

where  $C_1 \geq 0$  and  $m$  is the number of positive roots of  $G$ , and for all  $X \in \mathfrak{g}$ , there exists  $C_2 \geq 0$  so that

$$\|d\pi_\lambda(X)\|_{HS} \leq C |\lambda|^{\frac{m+2}{2}} |X|. \tag{5.5}$$

We will also need *Sugiura's zeta function*  $\zeta : \mathbb{C} \rightarrow \mathbb{R} \cup \{\infty\}$ , defined by

$$\zeta(s) = \sum_{\lambda \in D_0} \frac{1}{|\lambda|^s}, \tag{5.6}$$

which converges whenever  $2\Re(s) > r$ , where  $r$  is the rank of  $G$ . Finally we will need the fact that there is a topological isomorphism between  $C^\infty(G)$  and the Sugiura space of rapidly decreasing functions on  $D$ . We won't need full details of this, only the following fact that, for ease of reference, we will state as a theorem:

**Theorem 5.2.** *The function  $f \in C^\infty(G)$  if and only if for all  $p \geq 0$ ,*

$$\lim_{|\lambda| \rightarrow \infty} |\lambda|^p \|\widehat{f}(\lambda)\|_{HS} = 0,$$

where  $\widehat{f}(\lambda) := \widehat{f}(\pi_\lambda)$ .

**Theorem 5.3.** *There exist  $K = K(|X_1|, \dots, |X_d|) > 0$  so that for all  $\lambda \in D$ ,*

$$\sup_{\sigma \in G} \|j_A(\sigma, \lambda)\|_{HS} \leq K(1 + |\lambda|^{m+2}),$$

*Proof.* Throughout this proof  $K_1, K_2, \dots$  are non-negative constants. We examine each of the four terms on the right hand side of (5.1) in turn. For the first of these, we use (5.4) to obtain for all  $\lambda \in D$ ,

$$\|I_{\pi_\lambda}\|_{HS} = d_\lambda^{\frac{1}{2}} \leq K_1 |\lambda|^{m/2}.$$

For the second term we apply (5.5). For the third term, we employ the equivalence of norms in finite-dimensional vector spaces to find that for all  $X, Y \in \mathfrak{g}$ ,

$$\begin{aligned} \|d\pi_\lambda(X)d\pi_\lambda(Y)\|_{HS} &\leq K_2 \|d\pi_\lambda(X)d\pi_\lambda(Y)\|_{op} \\ &\leq K_2 \|d\pi_\lambda(X)\|_{op} \|d\pi_\lambda(Y)\|_{op} \\ &\leq K_3 \|d\pi_\lambda(X)\|_{HS} \|d\pi_\lambda(Y)\|_{HS} \\ &\leq K_4 |\lambda|^{m+2} |X| |Y|. \end{aligned}$$

For the term controlled by  $\nu$ , let  $V$  be a canonical neighbourhood of  $e$  for which  $x_1, \dots, x_d$  are canonical co-ordinates. Then for each  $\tau \in V$ ,

$$\pi_\lambda(\tau) = \pi_\lambda \left( \exp \left( \sum_{i=1}^d x_i(\tau) X_i \right) \right) = \exp \left( \sum_{i=1}^d x_i(\tau) d\pi_\lambda(X_i) \right),$$

where the final exp denotes the matrix exponential. Then by a standard Taylor series argument we have,

$$\begin{aligned} &\sup_{\sigma \in G} \left\| \int_V \left( \pi_\lambda(\tau) - I_{\pi_\lambda} - \sum_{i=1}^d x_i(\tau) d\pi_\lambda(X_i) \right) \nu(\sigma, d\tau) \right\|_{HS} \\ &\leq \sup_{\sigma \in G} \int_V \left\| \exp \left( \sum_{i=1}^d x_i(\tau) d\pi_\lambda(X_i) \right) - I_{\pi_\lambda} - \sum_{i=1}^d x_i(\tau) d\pi_\lambda(X_i) \right\|_{HS} \nu(\sigma, d\tau) \\ &\leq \sup_{\sigma \in G} \int_V \sum_{i,j=1}^d x_i(\tau) x_j(\tau) \|d\pi_\lambda(X_i) d\pi_\lambda(X_j)\|_{HS} \nu(\sigma, d\tau) \\ &\leq K_5 \max\{|X_1|^2, \dots, |X_d|^2\} |\lambda|^{m+2} \sup_{\sigma \in G} \int_V \sum_{i=1}^d x_i(\tau)^2 \nu(\sigma, d\tau) \\ &\leq K_6 |\lambda|^{m+2}. \end{aligned}$$

Finally we have

$$\begin{aligned}
& \sup_{\sigma \in G} \left\| \int_{V^c} \left( \pi_\lambda(\tau) - I_{\pi_\lambda} - \sum_{i=1}^d x_i(\tau) d\pi_\lambda(X_i) \right) \nu(\sigma, d\tau) \right\|_{HS} \\
& \leq \sup_{\sigma \in G} \int_{V^c} \left( 2d_\lambda^{\frac{1}{2}} + \sum_{i=1}^d |x_i(\tau)| \|d\pi_\lambda(X_i)\|_{HS} \right) \nu(\sigma, d\tau) \\
& \leq (2d_\lambda^{\frac{1}{2}} + K_7 \max_{i=1, \dots, d} \{ \|d\pi_\lambda(X_i)\|_{HS} \}) \sup_{\sigma \in G} \nu(\sigma, V^c) \\
& \leq K_8 (|\lambda|^{m/2} + |\lambda|^{(m+2)/2}).
\end{aligned}$$

The result follows on combining together all these estimates, bearing in mind that the functions  $c, b_i$  and  $a_{jk}$ , for  $1 \leq i, j, k \leq d$ , are all bounded on  $G$ .

□

**Lemma 5.4.** *If  $f \in C^\infty(G)$  then the series  $\sum_{\lambda \in D} d_\lambda \text{tr}(j_A(\sigma, \pi) \widehat{f}(\lambda) \pi_\lambda(\sigma))$  converges absolutely and uniformly in  $\sigma \in G$ .*

*Proof.* Using (5.4), Theorem 5.3, and the basic matrix estimate  $|\text{tr}(AB)| \leq \|A\|_{HS} \|B\|_{HS}$ , we have

$$\begin{aligned}
& \sum_{\lambda \in D} d_\lambda |\text{tr}(j_A(\sigma, \pi) \widehat{f}(\lambda) \pi_\lambda(\sigma))| \\
& \leq C_1 K \sum_{\lambda \in D} |\lambda|^m (1 + |\lambda|^{m+2}) \|\widehat{f}(\lambda)\|_{HS}.
\end{aligned}$$

By Theorem 5.2, we can find  $\lambda_0 \in D_0$  so that for all  $|\lambda| > |\lambda_0|$ ,

$$\|\widehat{f}(\lambda)\|_{HS} \leq \frac{B_1}{|\lambda|^s},$$

where  $s > 2(r + m + 1)$  and  $B_1 > 0$ . Then there exists  $B_2 > 0$  so that

$$\sup_{\sigma \in G} \sum_{|\lambda| > |\lambda_0|} d_\lambda |\text{tr}(j_A(\sigma, \pi) \widehat{f}(\lambda) \pi_\lambda(\sigma))| \leq B_2 \zeta(s - 2) < \infty.$$

□

**Theorem 5.5.** *The operator  $A : C^\infty(G) \rightarrow C(G)$  is a pseudo-differential operator with symbol  $j_A$ .*

*Proof.* We must show that for all  $f \in C^\infty(G)$ ,  $\sigma \in G$ ,

$$Af(\sigma) = \sum_{\pi \in \widehat{G}} d_\pi \operatorname{tr}(j_A(\sigma, \pi) \widehat{f}(\pi) \pi(\sigma)).$$

We have shown that this holds for  $f \in \mathcal{E}(G)$ . Now let  $f \in C^\infty(G)$ . By Theorem 3.3.1 (ii) of [5], the Fourier series  $\sum_{\lambda \in D} d_\lambda \operatorname{tr}(\widehat{f}(\lambda) \pi_\lambda(\sigma))$  converges absolutely and uniformly to  $f(\sigma)$ . Hence (by ordering the weights in  $D$  e.g. by increasing norm, with arbitrary ordering for those having the same norm) we obtain a sequence  $(f_n)$  of partial sums in  $\mathcal{E}(G)$  that converges uniformly to  $f$ . By lemma 5.4, we have the uniform convergence,

$$\lim_{n \rightarrow \infty} Af_n(\sigma) = \sum_{\pi \in \widehat{G}} d_\pi \operatorname{tr}(j_A(\sigma, \pi) \widehat{f}(\pi) \pi(\sigma)),$$

and the result follows since  $A$  is closed.  $\square$

Using techniques developed in [13], we believe that it should also be possible to represent operators satisfying the PMP as pseudo-differential operators when the group is homogeneous.

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