MAS331: Metric Spaces
Problems on Chapter 1

1. In $\mathbb{R}^3$, find $d_1((3,1,4),(2,7,1))$, $d_2((3,1,4),(2,7,1))$ and $d_\infty((3,1,4),(2,7,1))$.

2. In $\mathbb{R}^4$, show that $d_1((4,4,4,6),(0,0,0,0)) = d_1((3,5,5,5),(0,0,0,0))$ and $d_2((4,4,4,6),(0,0,0,0)) = d_2((3,5,5,5),(0,0,0,0))$.
Is $d_\infty((4,4,4,6),(0,0,0,0)) = d_\infty((3,5,5,5),(0,0,0,0))$?

3. Let $a = (a_1,\ldots,a_n), b = (b_1,\ldots,b_n) \in \mathbb{R}^n$.

   (a) Show that $d_\infty(a,b) \leqslant d_2(a,b) \leqslant d_1(a,b)$. (Hint: for the second inequality, first show that $d_1(a,b)^2 \geqslant d_2(a,b)^2$.)

   (b) Show that $d_1(a,b) \leqslant n d_\infty(a,b)$.

   (c) It follows from (a) and (b) that $d_2(a,b) \leqslant n d_\infty(a,b)$. Do better than this by showing that $d_2(a,b) \leqslant \sqrt{n} d_\infty(a,b)$.

   (d) When does $d_\infty(a,b) = d_2(a,b)$? When does $d_2(a,b) = d_1(a,b)$?

   (e) When does $d_1(a,b) = n d_\infty(a,b)$? When does $d_2(a,b) = \sqrt{n} d_\infty(a,b)$?

4. Prove that the *taxicab metric* d_1 on $\mathbb{R}^n$, given by

$$d_1((a_1,\ldots,a_n),(b_1,\ldots,b_n)) = |a_1 - b_1| + \cdots + |a_n - b_n|,$$

   is indeed a metric on $\mathbb{R}^2$.

5. Let $r > 0$ and let $a = (a_1,\ldots,a_n) \in \mathbb{R}^n$. Use Problem 3 to show that

$$B_1(a,r) \subseteq B_2(a,r) \subseteq B_\infty(a,r)$$

   and that

$$B_\infty(a,\frac{r}{n}) \subseteq B_1(a,r) \text{ and } B_\infty(a,\frac{r}{\sqrt{n}}) \subseteq B_2(a,r).$$

   Here, as in the notes, the subscripts indicate which metric we are using.

   When $n = 1$, all the inclusions $\subseteq$ are $=$. What can you say about them when $n > 1$?

6. Let $I = [0,5]$. In $C(I)$, let $f(x) = x^2 - 4x$ and $g(x) = 3x - 6$. Find $d_\infty(f,g)$ and $d_1(f,g)$.

7. Let $I = [0,1]$. In $C(I)$ with the metric d_∞, let f be the constant function with value 0, i.e. $f(x) = 0$ for all $x \in I$. Describe the closed ball $B[f,1]$.

8. Let X be any non-empty set. Show that the discrete metric satisfies all the axioms for a metric space.

9. (Exercise 1.19) Let $I = [a, b]$ and prove that $d_1 : C[a, b] \times C[a, b] \to \mathbb{R}$,

$$d_1(f, g) = \int_a^b |f(x) - g(x)| dx,$$

satisfies axioms M2 and M3.

10. There is a metric d on $\mathbb{R}^2$ such that, for $a, b \in \mathbb{R}^2$,

$$d(a,b) = \begin{cases} d_2(a,b) \text{ if } a, b \text{ and } (0,0) \text{ are collinear} \\ d_2(a,(0,0)) + d_2(b,(0,0)) \text{ otherwise.} \end{cases}$$

Thus $d(a, b)$ is the usual distance between a and b if we only allow movement along radii emanating from $(0, 0)$. (You are not asked to show that d is a metric. It is relevant in a country where all railway lines pass through the capital or a city where all bus routes go by the city hall.)

   (a) Find $d((4, 2), (2, 1))$ and $d((4, 2), (-2, 3))$.
   (b) Sketch the open balls $B((0, -1), \frac{1}{2})$ and $B((0, -1), 2)$.

11. Prove that the function d defined by $d(x, y) = \sqrt{|x - y|}$ (where $x, y \in \mathbb{R}$) is a metric on the set $\mathbb{R}$.

    Prove that the function d defined by $d(x, y) = (x - y)^2$ (where $x, y \in \mathbb{R}$) is *not* a metric on the set $\mathbb{R}$.

    [*Hint. To prove something is not a metric, you have only to show that one of the axioms doesn't hold, and to do this the best way is to give values of the variables for which it is false.*]

12. Let X be a non-empty set with two metrics d_1 and d_2. Define $d : X \times X \to \mathbb{R}$ by

$$d(x, y) = d_1(x, y) + d_2(x, y) \text{ for all } x, y \in X.$$

    Show that d is a metric on X.

13. In $\mathbb{R}_2$, let $d(x, y) = d_1(x, y) - d_2(x, y)$. Show that d is not a metric on $\mathbb{R}^2$.

14. For $a = (a_1, a_2, \cdots, a_n), b = (b_1, b_2, \cdots, b_n) \in \mathbb{R}^n$, let

$$e(a, b) = \max\{|a_i - b_i|\} + \min\{|a_i - b_i|\}.$$

   (a) Show that if $n = 2$ then, for all $a, b \in \mathbb{R}^n$, $e(a, b) = d_1(a, b)$, where d_1 is the taxicab metric.
   (b) By considering $(0, 0, 0), (1, 2, 3)$ and $(2, 2, 4)$, show that if $n = 3$ then e is not a metric.

MAS331: Metric Spaces
Problems on Chapter 2

1. Show that, in $\mathbb{R}$, the sequence $\sqrt{n+1}-\sqrt{n-1}$ converges to 0.

   Hint: Think of $\sqrt{n+1}-\sqrt{n-1}$ as a fraction $\frac{\sqrt{n+1}-\sqrt{n-1}}{1}$ and multiply top and bottom by $\sqrt{n+1}+\sqrt{n-1}$.

2. Show that $(\frac{n}{n+1}, \frac{n+1}{n}) \to (1,1)$ in the metric space $(\mathbb{R}^2, d_1)$, where d_1 is the "taxi-cab metric".

3. This question concerns the metrics d_1 and d_∞ on $\mathbb{R}^m$ and uses the notation $B_1(a,\varepsilon)$ and $B_\infty(a,\varepsilon)$ to distinguish between open balls in the two metrics. Let (a_n) be a sequence in $\mathbb{R}^m$ and let $a \in \mathbb{R}^m$. From Problem 5 on Chapter 1, we know that, for all $a \in \mathbb{R}^m$ and all $\varepsilon > 0$,

$$B_\infty(a, \varepsilon/m) \subseteq B_1(a,\varepsilon) \subseteq B_\infty(a,\varepsilon).$$

   Use this to show that $a_n \to a$ under d_1 if and only if $a_n \to a$ under d_∞. (The corresponding problem for d_2 and d_∞ will be on Assignment 2. Combining the two, if a sequence (a_n) converges to a under any one of d_1, d_2, d_∞ then it converges to a under all three.)

4. Show that the sequence $(x_n, y_n) = \left(\sin(\frac{1}{n^2+1}), \sqrt{\frac{1}{n^4+n}}\right)$ converges to $(0,0)$ in $\mathbb{R}^2$, with respect to each of the three metrics d_1, d_2 and d_∞. Hint: Use the fact that $|\sin x| \leqslant |x|$ for all real numbers x.

5. Show that the sequence (f_n) in $C[0,1]$, defined by

$$f_n(x) = x^2 + \frac{3x}{n} + \frac{2}{n^2},$$

   converges to the function $f(x) = x^2$ in both the d_1 and d_∞ metrics.

6. Show that the sequence (f_n) in $C[0,1]$, defined by

$$f_n(x) = \frac{n}{n+x},$$

   converges to the constant function $f(x) = 1$ in both the d_1 and d_∞ metrics. Hint: Recall that $(1+\frac{1}{n})^n \to e$ as $n \to \infty$.

7. Let (x_n) and (y_n) be two sequences in a metric space (X,d).

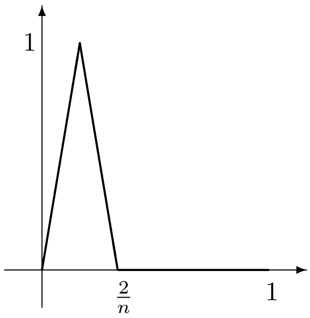
   (a) Show that if $x_n \to a$ and $y_n \to a$ then $d(x_n, y_n) \to 0$.

   (b) Show that if $x_n \to a$ and $d(x_n, y_n) \to 0$ then $y_n \to a$.

   (c) Show that if $x_n \to a$ and $y_n \to b$ then $d(x_n, y_n) \to d(a,b)$.
   [*Hint*: Use the identity $|d(x,y) - d(a,b)| \leqslant d(x,a) + d(y,b)$.]

8. Consider the sequence (f_n) of functions in $C[0,1]$ where

$$f_n(x) = \begin{cases} nx \text{ if } 0 \leqslant x \leqslant \frac{1}{n}, \\ 2 - nx \text{ if } \frac{1}{n} \leqslant x \leqslant \frac{2}{n}, \\ 0 \text{ if } \frac{2}{n} \leqslant x \leqslant 1. \end{cases}$$



(a) Show that f_n converges to 0 pointwise.

(b) Find $\int_0^1 f_n(x)\,dx$ (it's the area under the graph!).

(c) Prove that in the metric space $(C[0,1], d_1)$ we have $f_n \to 0$.

(d) Prove that in the metric space $(C[0,1], d_\infty)$ it is *not* true that $f_n \to 0$.

(e) Modify the example to get a sequence of functions (g_n) that converges pointwise to 0, but does not converge to 0 in the d_1 or d_∞ metric.
Hint: use a sequence of the form $g_n(x) = \lambda_n f_n(x)$, where λ_n is chosen carefully.

(f) Prove that in the metric space $(C[0,1]), d_\infty)$ the sequence (f_n) has no limit at all.

9. Let X be any non-empty set and let d be the discrete metric.

(a) Let $x \in X$. What is the open ball $B(x, \varepsilon)$ if $0 < \varepsilon < 1$? What is the open ball $B(x, \varepsilon)$ if $1 \leqslant \varepsilon$?

(b) Show that a sequence (x_n) converges to x in X if and only if it is eventually constant, that is there exists N such that for all $n > N$, $x_n = x$. Deduce that in $\mathbb{R}$ with the discrete metric the sequence $(\frac{1}{n})$ does not converge to 0.

MAS331: Metric Spaces
Problems on Chapter 3

1. In each of the following, write down a sequence (x_n) in the subset F of the metric space (X, d) that converges to a limit in $X \backslash F$. Deduce that F is not closed.

   (a) $X = \mathbb{R}$, $F = (\frac{1}{200}, \frac{1}{100})$.

   (b) $X = \mathbb{R}$, $F = [0, 1) \cup (1, 2]$.

   (c) $X = (\mathbb{R}^2, d_2)$, $F = \{(x, y) \in \mathbb{R}^2 : x + y \leqslant 0 \text{ and } x > 0\}$.

   (d) $X = (\mathbb{R}^2, d_\infty)$, $F = \{(x, y) \in \mathbb{R}^2 : 1 < d_\infty((x, y), (0, 0)) \leqslant 2\}$.

2. Show that the circle $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is a closed subset of $\mathbb{R}^2$ with the metric d_2.

3. Sketch the subset $A = \{(x, y) : x > 0 \text{ and } y > 0\}$ of the metric space $(\mathbb{R}^2, d_\infty)$, and sketch a typical small open ball around a point of A. (Note that we're using d_∞, not d_2!) Now prove that A is an open set in $(\mathbb{R}^2, d_\infty)$.

4. In the set $C[-1, 0]$, put $F = \{f \in C[-1, 0] : f(-1) = -1\}$.

   (a) Show that F is a closed subset of $(C[-1, 0], d_\infty)$.

   (b) For $n = 1, 2, 3, \ldots$, define $f_n \in C[-1, 0]$ by $f_n(x) = x^{2n} - 2$. Let g be the constant function on $[-1, 0]$, $g(x) = -2$ for all x. Show that $f_n \to g$ in $(C[-1, 0], d_1)$. Deduce that F is not a closed subset of $(C[-1, 0], d_1)$.

   (c) Show that (f_n) is not convergent in $(C[-1, 0], d_\infty)$.

5. Write down one example of a sequence (q_n) of rational numbers that converges, in $\mathbb{R}$, to an irrational limit λ. By considering the sequence $(\lambda - q_n)$, show that the set $\mathbb{I}$ of all irrational numbers is not closed in $\mathbb{R}$.

6. Show that the disc $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leqslant 5\}$ is a closed subset of $(\mathbb{R}^2, d_2)$ with its usual metric.

7. Let $a = (a_1, a_2 \ldots, a_m) \in \mathbb{R}^m$ and let $B = B(a, r)$ be an open ball in $\mathbb{R}^m$ for the metric d_2. By considering the sequence $(a_1 + r(1 - \frac{1}{n}), a_2, \ldots, a_m)$, show that B is not closed.

8. From Problem 5 on Chapter 1, we know that, for all $a \in \mathbb{R}^m$ and all $\varepsilon > 0$,

$$B_\infty(a, \varepsilon/m) \subseteq B_1(a, \varepsilon) \subseteq B_\infty(a, \varepsilon).$$

Let U be a subset of $\mathbb{R}^m$.

(a) Use Problem 3 on Chapter 2 to show that U is closed under d_1 if and only if U is closed under d_∞.

(b) Use Problem 5 on Chapter 1 to show that U is open under d_1 if and only if U is open under d_∞.

(c) Give an alternative proof of (b) using (a) and complements or an alternative proof of (a) using (b) and complements.

(The same results are true for d_2 and d_∞, see Assignment 2, so a subset of $\mathbb{R}^m$ is open (*resp* closed) under all three metrics d_1, d_2, d_∞ if it is open (*resp* closed) under any one of them.)

9. Let X be a metric space, let $x \in X$ and let F be the singleton set $\{x\}$. How many sequences with all their terms in F are there? Deduce that F is closed. Use a result from the notes to deduce that every finite subset of X is closed.

10. Let A be the annulus $\{(x, y) \in \mathbb{R} : 1 \leqslant x^2 + y^2 \leqslant 4\}$. Express A as the intersection of a closed ball and the complement of an open ball in $(\mathbb{R}^2, d_2)$ and deduce that A is closed in $(\mathbb{R}^2, d_2)$. Show also that $\{(x, y) \in \mathbb{R} : 1 < x^2 + y^2 < 4\}$ is open in $(\mathbb{R}^2, d_2)$.

11. Let X be a space with the discrete metric. Let $x \in X$. Show that $B[x, \frac{1}{2}] = B(x, \frac{1}{2}) = \{x\}$. Deduce that every subset Y of X is open. By considering complements, deduce that every subset Y of X is closed.

MAS331: Metric Spaces
Problems on Chapter 4

1. Consider $\mathbb{R}^2$ with the Euclidean metric and $\mathbb{R}$ with its usual metric.

   (a) Show that the function

   $$\begin{aligned} f\colon \mathbb{R}^2 &\to \mathbb{R} \\ (x,y) &\mapsto xy \end{aligned}$$

   is continuous at every point of $\mathbb{R}^2$.

   (b) Hence deduce that $\{(x,y) \in \mathbb{R}^2 \mid xy \geqslant 1\}$ is a closed subset of $\mathbb{R}^2$.

2. Let $c \in \mathbb{R}$ and let $f_c\colon (\mathbb{R}^2, d_2) \to \mathbb{R}$ be defined by

   $$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ c & \text{if } (x,y) = (0,0). \end{cases}$$

   - Show that f_c is continuous at any point $(x,y) \neq (0,0)$.
   - By considering the sequence $((\frac{1}{n}, \frac{k}{n}))$ for different values of $k \in \mathbb{R}$, show that f_c is not continuous at $(0,0)$.

3. Let (X, d_X) and (Y, d_Y) be metric spaces and let $\lambda > 0$. If $f\colon X \to Y$ satisfies $d_Y(f(x_1), f(x_2)) \leqslant \lambda d_X(x_1, x_2)$ for all $x_1, x_2 \in X$, show that f is continuous.

4. Let $f : \mathbb{R}^m \to \mathbb{R}^m$ be a function. Use Problem 3 on Chapter 2 to show that the following statements, about the continuity of f with respect to different combinations of metrics, are equivalent:

   (a) $f : (\mathbb{R}^m, d_1) \to (\mathbb{R}^m, d_1)$ is continuous;
   (b) $f : (\mathbb{R}^m, d_1) \to (\mathbb{R}^m, d_\infty)$ is continuous;
   (c) $f : (\mathbb{R}^m, d_\infty) \to (\mathbb{R}^m, d_\infty)$ is continuous;
   (d) $f : (\mathbb{R}^m, d_\infty) \to (\mathbb{R}^m, d_1)$ is continuous.

   (Using Assignment 2, Q4(i), this can be extended to combinations involving d_2, giving nine equivalent statements!)

5. Let (X, d) be any metric space and let $f : C[0,1] \to X$ and $g : X \to C[0,1]$ be functions. Using Proposition 2.15, show that if f is continuous when $C[0,1]$ has the metric d_1 then it is continuous when $C[0,1]$ has the metric d_∞ and that if g is continuous when $C[0,1]$ has the metric d_∞ then it is continuous when $C[0,1]$ has the metric d_1.

6. Let θ be the identity function from $C[0,1]$ to itself. Thus $\theta(f) = f$ for all $f \in C[0,1]$. Show that, as a function from $(C[0,1], d_1)$ to $(C[0,1], d_\infty)$, θ is not continuous but that, as a function from $(C[0,1], d_\infty)$ to $(C[0,1], d_1)$, θ is continuous. (Use the sequential definition of continuity together with one example and one result in the notes from Chapter 2.)

7. Let (X, d_X), (Y, d_Y), (Z, d_Z) be metric spaces.

   (a) Show that if $y \in Y$ then the constant function $f\colon X \to Y$ given by $f(x) = y$ for all $x \in X$ is continuous.

   (b) Show that if $f\colon X \to Y$, $g\colon Y \to Z$ are continuous then the composite $g \circ f\colon X \to Z$ is continuous.

   (c) Give a second proof of (b) using whichever of the two definitions of continuity you didn't use in (b).

8. Define $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = 0$ if $x \neq 0$, and $f(0) = 1$. Show that f is not continuous by finding a closed subset $A \subseteq \mathbb{R}$ such that $f^{-1}(A)$ is not closed.

9. In Problem 10 on Chapter 3, it was shown that $A = \{(x, y) \in \mathbb{R} : 1 \leqslant x^2 + y^2 \leqslant 4\}$ is closed in $(\mathbb{R}^2, d_2)$ and that $C = \{(x, y) \in \mathbb{R} : 1 < x^2 + y^2 < 4\}$ is open in $(\mathbb{R}^2, d_2)$. Give alternative proofs for these facts by expressing A and C as inverse images of intervals for an appropriate continuous function from $(\mathbb{R}^2, d_2)$ to $\mathbb{R}$.

MAS331: Metric Spaces
Problems on Chapter 5

1. Let (f_n) be the sequence in $(C[0,1], d_\infty)$ such that, for each $x \in [0,1]$,

$$f_n(x) = 1 + \frac{x}{2} + \frac{x^2}{2^2} + \cdots + \frac{x^n}{2^n}.$$

   (a) For $m > n$, show that $f_m(x) - f_n(x) \geqslant 0$ for all $x \in [0,1]$ and that $f_m - f_n$ is strictly increasing on $(0,1]$. Hence show that, for $m > n$,

$$d_\infty(f_n, f_m) = f_m(1) - f_n(1) = \frac{1}{2^n} - \frac{1}{2^m}.$$

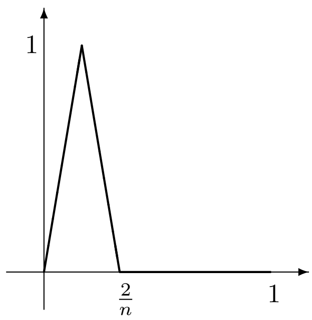
   (b) Show that (f_n) is a Cauchy sequence.

   (c) Does (f_n) converge in $(C[0,1], d_\infty)$? If so, why?

2. Recall, from Problem 8 on Chapter 2, the sequence (f_n) of functions in $C[0,1]$ such that

$$f_n(x) = \begin{cases} nx \text{ if } 0 \leqslant x \leqslant \frac{1}{n}, \\ 2 - nx \text{ if } \frac{1}{n} \leqslant x \leqslant \frac{2}{n}, \\ 0 \text{ if } \frac{2}{n} \leqslant x \leqslant 1. \end{cases}$$

   whose graphs are as shown in the diagram.



   Let $n \geqslant 1$ and let $x = \frac{1}{n}$. Compute $f_n(x)$ and $f_{2n}(x)$. Deduce that $d_\infty(f_{2n}, f_n) \geq 1$ and hence that (f_n) is not Cauchy in $(C[0,1], d_\infty)$. Show also that if $m \geqslant 2n$ then $d_\infty(f_m, f_n) \geq 1$ and that no subsequence of (f_n) is Cauchy in $(C[0,1], d_\infty)$.

3. Give an example of a Cauchy sequence in the unbounded open interval $(0, \infty)$ that is not convergent in $(0, \infty)$. Give an example of a Cauchy sequence in the set $\mathbb{I}$ of all irrational numbers that is not convergent in $\mathbb{I}$. Deduce that $(0, \infty)$ and $\mathbb{I}$ are not complete.

4. Let X be a set equipped with two metrics, d and d', and suppose that there are constants $J, K > 0$ such that

$$Jd'(x,y) \leqslant d(x,y) \leqslant Kd'(x,y)$$

for all $x, y \in X$. (For example, X could be $\mathbb{R}^m$ and d and d' could be any two of d_1, d_2 and d_∞, see Problem 3 on Chapter 1.)

   (a) Show that $K^{-1}d(x,y) \leqslant d'(x,y) \leqslant J^{-1}d(x,y)$ for all $x, y \in X$. (This is easy and makes the "if and only if" proofs in the later parts symmetric.)

   (b) Show that $x_n \to x$ in the metric space (X, d) if and only if $x_n \to x$ in the metric space (X, d').

   (c) Show that (x_n) is a Cauchy sequence in (X, d) if and only if it is a Cauchy sequence in (X, d').

   (d) Show that (X, d) is complete if and only if (X, d') is complete.

5. Use Problem 4 together with Problem 3 on Chapter 1 and Theorem 5.10 to show that, for $m \geqslant 1$, $\mathbb{R}^m$ is complete with respect to the metric d_∞ and with respect to the metric d_1.

6. Let $X = (0, \infty)$ viewed as a subspace of $\mathbb{R}$ with its usual metric. Give an example of a Cauchy sequence (x_n) in X and a continuous function $f : X \to X$ such that $(f(x_n))$ is not a Cauchy sequence. (This is in contrast to the situation for convergent sequences, where a continuous function sends convergent sequences to convergent sequences.)

7. Let X be a non-empty set and let d be the discrete metric on X. Let (x_n) be a Cauchy sequence in (X, d). Show that there exist $N \in \mathbb{R}$ and $a \in X$ such that $x_n = a$ for all $n > N$. Deduce that (X, d) is complete.

MAS331: Metric Spaces
Problems on Chapter 6

1. (a) Consider the function $f\colon [1,\infty) \to [1,\infty)$ defined by $f(x) = x + \frac{1}{x}$. Prove that $|f(x) - f(y)| < |x - y|$ whenever $x \neq y$, but that f has no fixed point. (done in class this year)

   (b) Fix $\frac{1}{2} \leqslant k < 1$ and let $g(x) = kf(x)$. Show that $g(1) \geq 1$ and that g is increasing on $(1,\infty)$. Deduce that $g(x) \geq 1$ for all $x \in [1,\infty)$ so that we can view g as a function from $[1,\infty)$ to $[1,\infty)$.

   (c) Show that if k and g are as in (b), then g has a unique fixed point in $[1,\infty)$. Find the fixed point when $k = \frac{3}{4}$.

   (d) Taking $x_0 = 1$, and iterating with $x_{n+1} = \frac{3}{4}(x_n + \frac{1}{x_n})$, show that $|x_n - x| \leq \frac{2\times 3^n}{4^n}$.

2. Prove that the function $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$f(x,y,z) = (\tfrac{1}{2}\cos y + 1, \tfrac{2}{3}\sin z, \tfrac{3}{4}x)$$

   is a contraction, where $\mathbb{R}^3$ is given the Euclidean metric.

3. Let $x, y \in [-1,1]$. Show that $|\sin(\frac{x+y}{2})| \leqslant \sin(1)$. Hence show that the function $f : [-1,1] \to [-1,1]$ given by $f(x) = \cos(x)$ is a contraction on $[-1,1]$ with contraction factor $\sin(1)$.

4. Let $f\colon [-1,1] \to [-1,1]$ be defined by $f(x) = \frac{1}{6}(x^3 + x^2 + 1)$. Use the differential criterion to show that f is a contraction of $[-1,1]$. Quote two results from Chapter 5 to explain why $[-1,1]$ is complete. Deduce that there is a unique value of $x \in [-1,1]$ such that $x^3 + x^2 - 6x + 1 = 0$.

   (To see that $f\colon [-1,1] \to [-1,1]$, use elementary calculus: f has stationary points at $\frac{-2}{3}$ and 0. $f(-1) = f(0) = \frac{1}{6} \in [-1,1]$, $f(\frac{-2}{3}) = \frac{31}{27\times 6} \in= f(0)$ and $f(1) = \frac{1}{2} \in [-1,1]$ so $f : [-1,1] \to [-1,1]$.)

5. Use the differential criterion to show that $\cos \circ \sin$ is a contraction on $\mathbb{R}$ with contraction factor $k = \sin(1)$. Deduce that there is a unique element $x \in \mathbb{R}$ such that $\cos(\sin(x)) = x$.

6. Show that the function $h\colon [1,\infty) \to [1,\infty)$ defined by $h(x) = 1 + e^{-x}$ is a contraction with contraction factor e^{-1}. Deduce that the equation

$$x = 1 + e^{-x}$$

   has a unique solution in $[1,\infty)$.

7. Let f and g both be contractions of the metric space (X,d) with contraction factors k and k' respectively. Show that the function $f \circ g$ (which takes x to $f(g(x))$) is a contraction of (X,d) with contraction factor kk'.

   Show also that if x is a fixed point of $f \circ g$ then $g(f(g(x))) = g(x)$. Hence find a fixed point of $g \circ f$ (in terms of x).

8. Let $T\colon C[0,\frac{1}{2}] \to C[0,\frac{1}{2}]$ be given by

$$(T(f))(x) = 1 + \int_0^x f(u)\ du.$$

(Given any $f \in C[0,\frac{1}{2}]$ this formula enables you to work out $T(f)$ which, naturally, is another member of $C[0,\frac{1}{2}]$. For example, if $f(x) = x^2$ then

$$(T(f))(x) = 1 + \int_0^x u^2\, du = 1 + \frac{x^3}{3}.\Big).$$

(a) Calculate $(T(f))(x)$ when $f(x) = \sin x$.

(b) Show that for any $f, g \in C[0,\frac{1}{2}]$ and $x \in [0,\frac{1}{2}]$

$$|(T(f))(x) - (T(g))(x)| = \left|\int_0^x (f(u) - g(u))du\right| \leqslant \tfrac{1}{2} d_\infty(f,g)$$

and deduce that $d_\infty(T(f),T(g)) \leqslant \frac{1}{2} d_\infty(f,g)$, so that T is a contraction.

(c) Let the sequence $f_1, f_2, f_3, \ldots$ in $C[0,\frac{1}{2}]$ be defined by $f_1(x) = 1 + x$, $f_2 = T(f_1)$, $f_3 = T(f_2), \ldots$. Calculate $f_2(x)$ and $f_3(x)$. Deduce (without proof) a formula for $f_n(x)$. State, without proof, the limit $f(x)$ of this sequence, giving your answer as a standard function.

(d) Our general theory tells us that the f found in (c) should be the unique fixed point of T; i.e. it should satisfy

$$f(x) = 1 + \int_0^x f(u)\ du.$$

Verify that the function f found in (c) does satisfy this equation.

9. Let $T\colon C[0,1] \to C[0,1]$ be given by

$$(T(f))(x) = 1 + 3\int_0^x u^2 f(u)\, du.$$

(a) Starting with f_1 given by $f_1(x) = 1$, iterate T to find the next three terms of the sequence $f_2 = T(f_1)$, $f_3 = T(f_2), \ldots$. Guess (in series form) the limit $f(x)$ of this sequence. Then express $f(x)$ in terms of the exponential function.

(b) Show, by working out the right-hand side, that the f found in (a) is a solution of the integral equation

$$f(x) = 1 + 3\int_0^x u^2 f(u)\, du.$$

(c) Show by differentiating f that it is a solution of the differential equation

$$\frac{df}{dx} = 3x^2 f \qquad\qquad (x \in [0,1])$$

satisfying the initial condition $f(0) = 1$.

10. Let (x_n) be a Cauchy sequence in a complete metric space X, with limit x. Show that, for all m, n,

$$\begin{aligned} d(x_m, x_n) &\leqslant d(x_m, x) + d(x, x_n) \text{ and} \\ d(x, x_n) &\leqslant d(x, x_m) + d(x_m, x_n). \end{aligned}$$

Deduce that

$$-d(x, x_m) \leq d(x_m, x_n) - d(x, x_n) \leq d(x, x_m)$$

and hence that, if n is fixed, $d(x_m, x_n) \to d(x, x_n)$ as $m \to \infty$.

MAS331: Metric Spaces
Problems on Chapter 7

1. For each of the following subsets K of $\mathbb{R}^2$, state whether K is compact, justifying your answers.

   (a) $\{(x, y) \in \mathbb{R}^2 : x \geqslant 0 \text{ and } y \geqslant 0\}$;

   (b) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$;

   (c) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \geq 1\}$;

   (d) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

2. Let A be the subset

$$\{(x, y) \in \mathbb{R}^2 : 1 \leqslant (x - 4)^2 + (y - 4)^2 \leqslant 4\}$$

   of $\mathbb{R}^2$. Write down the smallest positive real number D such that $d_2(a, b) \leqslant D$ for all $(a, b) \in A$ and the smallest positive real number M such that $d_2(a, 0) \leqslant M$ for all $(a, b) \in A$.

3. Let A be a subset of $\mathbb{R}^m$ for some $m \geqslant 1$. Using Problem 3 on Chapter 2 and Problem 4(i) from Assignment 2, or otherwise, show that if A is compact under any one of the metrics d_1, d_2, d_∞ then it is compact under all three.

4. Show that a finite union of compact sets in a metric space is again compact.

5. Show that a set X with the discrete metric is compact precisely when X is a finite set.

6. Let X be a metric space, let A be a subset of X and let $x \in X$. Show that A is bounded if and only if there exists M such that $d(a, x) \leqslant M$ for all $a \in A$.

7. Let X be a compact metric space and let $F_1 \supset F_2 \supset \ldots \supset F_n \supset F_{n+1} \supset \ldots$ be a sequence of non-empty closed subsets. Show that $F_1 \cap F_2 \cap \cdots \cap F_n \cap \cdots$ is non-empty.

8. Let X be a compact metric space and let $f \colon X \to X$ be a continuous function without any fixed points (that is, $f(x) \neq x$ for any $x \in X$). Show that there is an $\epsilon > 0$ such that $d(f(x), x) \geqslant \epsilon$ for all $x \in X$.

9. Let X be a compact metric space and let $f \colon X \to X$ be a continuous function. Show that there is a non-empty closed subset $A \subset X$ such that $f(A) = A$.

   Hint: Consider $F_1 = f(X)$, $F_2 = f(F_1), \ldots$ *and* $A = \bigcap F_n$.