The Relationship Between the Riemann and Lebesgue Integrals

In this section, our aim is to show that if a bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable, then it is measurable and Lebesgue integrable. Moreover the Riemann and Lebesgue integrals coincide. We begin by briefly revising the Riemann integral.

(1) The Riemann Integral

A partition $\mathcal{P}$ of $[a, b]$ is a set of points $\{x_0, x_1, \ldots, x_n\}$ with $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. Define $m_j = \inf_{x_{j-1} \le x \le x_j} f(x)$ and $M_j = \sup_{x_{j-1} \le x \le x_j} f(x)$. We underestimate by defining

$$L(f, \mathcal{P}) = \sum_{j=1}^{n} m_j (x_j - x_{j-1}),$$

and overestimate by defining

$$U(f, \mathcal{P}) = \sum_{j=1}^{n} M_j (x_j - x_{j-1}),$$

A partition $\mathcal{P}'$ is said to be a *refinement* of $\mathcal{P}$ if $\mathcal{P} \subset \mathcal{P}'$. We then have

$$L(f, \mathcal{P}) \le L(f, \mathcal{P}'), \quad U(f, \mathcal{P}') \le U(f, \mathcal{P}). \tag{0.0.1}$$

A sequence of partitions $(\mathcal{P}_n)$ is said to be *increasing* if $\mathcal{P}_{n+1}$ is a refinement of $\mathcal{P}_n$ for all $n \in \mathbb{N}$.

Now define the *lower integral* $L_{a,b} f = \sup_{\mathcal{P}} L(f, \mathcal{P})$, and the *upper integral* $U_{a,b} f = \inf_{\mathcal{P}} U(f, \mathcal{P})$. We say that f is *Riemann integrable* over $[a, b]$ if $L_{a,b} f = U_{a,b} f$, and we then write the common value as $\int_a^b f(x)dx$. In particular, every continuous function on $[a, b]$ is Riemann integrable. The next result is very useful:

Theorem 0.0.1 *The bounded function f is Riemann integrable on $[a, b]$ if and only if for every $\epsilon > 0$ there exists a partition $\mathcal{P}$ for which*

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) < \epsilon. \tag{0.0.2}$$

If (0.0.2) holds for some $\mathcal{P}$, it also holds for all refinements of $\mathcal{P}$. A useful corollary is

Corollary 0.0.1 *If the bounded function f is Riemann integrable on $[a,b]$, then there exists an increasing sequence $(\mathcal{P}_n)$ of partitions of $[a,b]$ for which*

$$\lim_{n\to\infty} U(f,\mathcal{P}_n) = \lim_{n\to\infty} L(f,\mathcal{P}_n) = \int_a^b f(x)dx$$

Proof. This follows from Theorem (0.0.1) by successively choosing $\epsilon = 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$. If the sequence $(\mathcal{P}_n)$ is not increasing, then just replace $\mathcal{P}_n$ with $\mathcal{P}_n \cup \mathcal{P}_{n-1}$ and observe that this can only improve the inequality (0.0.2). □

(2) The Connection

Theorem 0.0.2 *If $f : [a,b] \to \mathbb{R}$ is Riemann integrable, then it is Lebesgue integrable, and the two integrals coincide.*

Proof. We use the notation λ for Lebsgue measure in this section. We also write $M = \sup_{x\in[a,b]} |f(x)|$ and $m = \inf_{x\in[a,b]} |f(x)|$.

Let $\mathcal{P}$ be a partition as above and define simple functions,

$$g_{\mathcal{P}} = \sum_{j=1}^{n} m_j \mathbf{1}_{(x_{j-1},x_j]}, \quad h_{\mathcal{P}} = \sum_{j=1}^{n} M_j \mathbf{1}_{(x_{j-1},x_j]}.$$

Consider the sequences (g_n) and (h_n) which correspond to the partitions of Corollary 0.0.1 and note that

$$L_n(f) = \int_{[a,b]} g_n d\lambda, \quad U_n f = \int_{[a,b]} h_n d\lambda,$$

where $U_n(f) = U(f,\mathcal{P}_n)$ and $L_n(f) = L(f,\mathcal{P}_n)$. Clearly we also have for each $n \in \mathbb{N}$,

$$g_n \leq f \leq h_n. \tag{0.0.3}$$

Since (g_n) is increasing (by (0.0.1)) and bounded above by M, it converges pointwise to a measurable function g. Similarly (h_n) is decreasing and bounded below by m, so it converges pointwise to a measurable function h. By (0.0.3) we have

$$g \leq f \leq h. \tag{0.0.4}$$

Again since $\max_{n\in\mathbb{N}}\{|g_n|, |h_n|\} \leq M$, we can use dominated convergence to deduce that g and h are both integrable on $[a,b]$ and by Corollary 0.0.1,

$$\int_{[a,b]} g d\lambda = \lim_{n\to\infty} L_n(f) = \int_a^b f(x)dx = \lim_{n\to\infty} U_n(f) = \int_{[a,b]} h d\lambda.$$

Hence we have

$$\int_{[a,b]} (h - g) d\lambda = 0,$$

and so by Corollary 3.3.1, $h(x) = g(x)$ (a.e.). Then by (0.0.4) $f = g$ (a.e.) and so f is measurable[1] and also integrable. So $\int_{[a,b]} f d\lambda = \int_{[a,b]} g d\lambda$, and hence we have

$$\int_{[a,b]} f d\lambda = \int_a^b f(x) dx. \qquad \square$$

[1] I'm glossing over a subtlety here. It is not true in general, that a function that is almost everywhere equal to a measurable function is measurable. It works in this case due to a special property of Lebesgue measure called "completeness."