# SECOND QUANTISATION FOR SKEW CONVOLUTION PRODUCTS OF INFINITELY DIVISIBLE MEASURES

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Abstract. Suppose $\lambda_1$ and $\lambda_2$ are infinitely divisible Radon measures on real Banach spaces $E_1$ and $E_2$, respectively and let $T : E_1 \to E_2$ be a Borel measurable mapping so that $T(\lambda_1) * \rho = \lambda_2$ for some Radon probability measure $\rho$ on $E_2$. Extending previous results for the Gaussian and the Poissonian case, we study the problem of representing the 'transition operator' $P_T : L^p(E_2, \lambda_2) \to L^p(E_1, \lambda_1)$ given by

$$P_T f(x) = \int_{E_2} f(T(x) + y)d\rho(y)$$

as the second quantisation of a contraction operator acting between suitably chosen 'reproducing kernel Hilbert spaces' associated with $\lambda_1$ and $\lambda_2$.

## 1. Introduction

Let $E_i$ $(i = 1, 2)$ be real Banach spaces equipped with Radon probability measures $\lambda_1$ and $\lambda_2$, respectively. A Borel measurable mapping $T : E_1 \to E_2$ is called a *skew map* for the pair $(\lambda_1, \lambda_2)$ if there exists a Radon probability measure $\rho$ on $E_2$ so that $\lambda_2$ is the convolution of $\rho$ with the image of $\lambda_1$ under the action of $T$:

$$T(\lambda_1) * \rho = \lambda_2.$$

In this case for each $1 \le p < \infty$ we obtain a linear contraction $P_T : L^p(E_2, \lambda_2) \to L^p(E_1, \lambda_1)$ given by

$$P_T f(x) = \int_{E_2} f(T(x) + y)d\rho(y).$$

Such constructions arise naturally in the study of Mehler semigroups, linear stochastic partial differential equations driven by additive Lévy noise, and operator self-decomposable measures (see, e.g., [4, 6, 7, 9, 10, 17]). In this context, the problem of "second quantisation" is to find a functorial manner of expressing $P_T$ in terms of $T$. The reason for this name is that the first work on this subject [5], within the context of Gaussian measures, exploited constructions that were similar to those that are encountered in the construction of the free quantum field from one-particle space (see e.g. [14]) wherein the $n$th chaos spanned by multiple Wiener-Itô integrals corresponds to the $n$-particle space within the Fock space decomposition.

*Date*: August 5, 2014.

2000 *Mathematics Subject Classification.* Primary 60B11, Secondary 60E07, 60G51, 60G57, 60H05.

*Key words and phrases.* Second quantisation, infinitely divisible measure, Wiener-Itô decomposition, Poisson random measure.

The second named author was supported by VICI subsidy 639.033.604 of the Netherlands Organisation for Scientific Research (NWO).

In our previous paper [3] we implemented this programme and constructed $P_T$ as the second quantisation of $T$ in the two cases where the measure $\lambda_i$ are Gaussian (generalising [5] and [13]), and are infinitely divisible measures of pure jump type (generalising [15]). In this article, we complete the programme by dealing with the case where the $\lambda_i$ are general infinitely divisible measures. Recall that a Radon probability measure $\lambda$ on $E$ is said to be *infinitely divisible* of for each integer $n \geq 1$ there exists a Radon probability measure $\lambda_{1/n}$ whose $n$-fold convolution product equals $\lambda$:

$$\underbrace{\lambda_{1/n} * \ldots * \lambda_{1/n}}_{n \text{ times}} = \lambda.$$

These measures $\lambda_{1/n}$ are unique.

It is well-known that an infinitely divisible Radon probability measure $\lambda$ on $E$ admits a unique representation as the convolution

$$\lambda = \delta_\xi * \gamma * \tilde{e}_{\mathrm{s}}(\nu), \tag{1.1}$$

where $\delta_\xi$ is the Dirac measure concentrated at the point $\xi \in E$, $\gamma$ is a centred Gaussian Radon measure on $E$, and $\tilde{e}_{\mathrm{s}}(\nu)$ is the generalised exponential of a Radon Lévy measure $\nu$ on $E$ as in [8, Theorem 3.4.20].

It is useful to rewrite (1.1) from the point of view of random variables, rather than measures. By [8, Theorem 2.39] there exists a semigroup of Radon probability measures $(\lambda_t)_{t\geq 0}$ such that $\lambda = \lambda_1$. By the celebrated Kolmogorov construction (see, e.g., [1, pp. 64–5]) we may construct an $E$-valued process $(X_t)_{t\geq 0}$ such that the law of $X_t$ is $\lambda_t$ for each $t \geq 0$. Using the Lévy-Itô decomposition of Riedle and van Gaans [16], for $t = 1$ we may then write

$$X_1 = \xi + Q + \int_E x\, d\bar{\Pi}(x),$$

where $\xi \in E$ is as in (1.1), $Q$ is the covariance of $\gamma$, and $\Pi$ is a Poisson random measure whose intensity measure $\nu$ is a Lévy measure on $E$ and

$$\bar{\Pi}(dx) := \mathbf{1}_{\{0<\|x\|\leq 1\}}\widehat{\Pi}(dx) + \mathbf{1}_{\{\|x\|>1\}}\Pi(dx),$$

with $\widehat{\Pi}$ the compensated Poisson random measure,

$$\widehat{\Pi}(B) := \Pi(B) - \nu(B).$$

In this description, the measure $\tilde{e}_{\mathrm{s}}(\nu)$ is the law of $\int_E x\, d\bar{\Pi}(x)$.

The data $\xi$, $\gamma$, $\nu$ are uniquely determined by $\lambda$. For more details we refer to [3, 12, 15, 16]. In what follows we shall write

$$\pi := \delta_\xi * \tilde{e}_{\mathrm{s}}(\nu) \tag{1.2}$$

for brevity.

From [3], we know that we can effectively realise the second quantisation of skew maps of $\gamma$ in the symmetric Fock space $\Gamma(H_\gamma)$ of the reproducing kernel Hilbert space $H_\gamma$ of $\gamma$; by the Wiener-Itô chaos decomposition this space is isomorphic to $L^2(E,\gamma)$. To second quantise skew maps of $\pi$, we use the fact that a similar result holds if instead of the symmetric Fock space over $H_\gamma$, we consider the symmetric Fock space over $L^2(E,\nu)$; this is precisely the approach adopted by Peszat in [15].

The independence of $X_\gamma$ and $X_\pi$ then suggests that in order to unify these two approaches one should use the symmetric Fock space over $H_\gamma \oplus L^2(E,\nu)$. As we shall demonstrate in this paper, this intuition is correct.

We finish this introduction by fixing some notation. All vector spaces are real. Unless otherwise stated, Banach spaces are denoted by $E$, $F, \dots$, and Hilbert spaces by $H$. The dual of a Banach space $E$ is denoted by $E^*$; the duality pairing between vectors $x \in E$ and $x^* \in E^*$ is written as $\langle x, x^* \rangle$. Using the Riesz representation theorem, the dual of a Hilbert space $H$ will always be identified with $H$ itself. The Fourier transform of a Radon probability measure $\mu$ defined on $E$ is the mapping $\widehat{\mu} : E^* \to \mathbb{C}$ for which

$$\widehat{\mu}(x^*) = \int_E e^{i\langle x, x^* \rangle}\, d\mu(x).$$

## 2. Skew Convolution Products of Infinitely Divisible Measures

We fix two infinitely divisible Radon probability measures $\lambda_1$ and $\lambda_2$, on the Banach spaces $E_1$ and $E_2$ respectively. We furthermore assume that a Borel linear mapping $T : E_1 \to E_2$ is given. The main result of this section gives a necessary and sufficient condition in order that $T$ be skew with respect to the pair $(\lambda_1, \lambda_2)$.

We recall the Lévy-Khintchine decompositions $\lambda_i = \gamma_i * \pi_i$ of (1.1) and (1.2) (for $i = 1, 2$)

**Proposition 2.1.** *Under these assumptions the following assertions are equivalent:*

1. *$T$ is skew with respect to $(\lambda_1, \lambda_2)$ with an infinitely divisible skew factor;*
2. *$T$ is skew with respect to both $(\gamma_1, \gamma_2)$ and $(\pi_1, \pi_2)$ with infinitely divisible skew factors.*

*If these equivalent conditions are satisfied, the skew factor $\rho$ in* (1) *and the skew factors $\rho_\gamma$ and $\rho_\pi$ in* (2) *are related by $\rho = \rho_\gamma * \rho_\pi$.*

*Proof.* We begin by making the preliminary observation that if $\alpha$ and $\beta$ are measures on $E_1$, then their image measures under $T$ satisfy $T(\alpha * \beta) = (T\alpha) * (T\beta)$. We shall freely use the properties of infinitely divisible measures on Banach space as can be found in [8, 12].

(2)⇒(1): From

$$T\lambda_1 * (\rho_\gamma * \rho_\pi) = (T\gamma_1 * T\pi_1) * (\rho_\gamma * \rho_\pi) = (T\gamma_1 * \rho_\gamma) * (T\pi_1 * \rho_\pi) = \gamma_2 * \pi_2 = \lambda_2$$

we infer that $T$ is skew for $(\lambda_1, \lambda_2)$ with skew factor $\rho_\gamma * \rho_\pi$. This measure, being the convolution of two infinitely divisible measures, is infinitely divisible.

(1)⇒(2): By the Lévy-Khintchine decomposition theorem we have $\lambda_i = \delta_{x_i} * \gamma_i * \tilde{e}_{\mathrm{s}}(\nu_i)$ $(i = 1, 2)$ using the notation introduced before we have

$$\lambda_1 * \lambda_2 = (\delta_{\xi_1} * \gamma_1 * \tilde{e}_{\mathrm{s}}(\nu_1)) * (\delta_{\xi_2} * \gamma_2 * \tilde{e}_{\mathrm{s}}(\nu_2)) = \delta_{\xi_1 + \xi_2} * (\gamma_i * \gamma_2) * \tilde{e}_{\mathrm{s}}(\nu_i + \nu_2).$$

By the uniqueness part of [8, Theorem 3.4.20], this shows that the Gaussian factor of $\lambda_1 * \lambda_2$ equals $\gamma_1 * \gamma_2$.

Now suppose that $T\lambda_1 * \rho = \lambda_2$ with each of the measures $\lambda_1, \lambda_2$, and $\rho$ infinitely divisible. Then $T\lambda_1$ is infinitely divisible with $T\lambda_1 = \delta_{Tx_1} * T\gamma_1 * T\tilde{\varepsilon}_{\mathrm{s}}(\nu_1)$, and applying the remark of the previous paragraph to $T\lambda_1$ and $\rho$ we find that the Gaussian factor of $T\lambda_1 * \rho$ equals $T\gamma_1 * \eta$, where $\eta$ is the Gaussian factor of $\rho$. It follows that

$$T\gamma_1 * \eta = \gamma_2,$$

that is, $T$ is skew with respect to $(\gamma_1, \gamma_2)$ with Gaussian factor $\eta$. Taking Fourier transforms, this means that

$$\widehat{T\gamma_1}\,\widehat{\eta} = \widehat{\gamma}_2. \tag{2.1}$$

Finally, taking Fourier transforms in the original identity $T\lambda_1 * \rho = \lambda_2$ we obtain $\widehat{T\gamma_1}\widehat{T\pi_1}\widehat{\rho} = \widehat{\gamma_2}\widehat{\pi_2}$ or equivalently, utilising (2.1)

$$\widehat{T\pi_1}\Big(\frac{\widehat{T\gamma_1}}{\widehat{\gamma_2}}\widehat{\rho}\Big) = \widehat{T\pi_1}\widehat{\eta}\widehat{\rho} = \widehat{T\pi_1}\widehat{\eta * \rho} = \widehat{\pi_2}.$$

From this we see that $T$ is skew with respect to $(\pi_1, \pi_2)$, with skew factor $\eta * \rho$. $\square$

It is not true in general that $\mu_1 * \mu_2 = \mu_3$ with $\mu_1$ and $\mu_3$ infinite divisible implies the infinite divisibility of $\mu_2$. The following counterexample (in the case $E = \mathbb{R}$) is due to Jan Rosiński who kindly kindly permitted its inclusion here.

*Example* 2.2 (Rosiński). Consider the signed measure $\nu := 2\delta_1 + 2\delta_2 - \delta_3 + 2\delta_4 + 2\delta_5$, where $\delta_x$ is the usual Dirac mass at $x \in \mathbb{R}$. We claim that

$$\phi(t) := \exp\Big(\int_0^\infty (e^{itx} - 1)\, d\nu(x)\Big)$$

is the characteristic function of some non-negative random variable $Z$. This random variable cannot be infinitely divisible. Indeed, if it were, $\nu$ would be its Lévy measure, which is impossible because a Lévy measure is non-negative and unique. Therefore, to complete a counterexample we need to show that $\phi$ is a characteristic function. Consider

$$e(\nu) := \sum_{n=0}^{\infty} \frac{\nu^{*n}}{n!}.$$

First we compute

$$\nu^{*2} = 4\delta_2 + 8\delta_3 + 4\delta_5 + 17\delta_6 + 4\delta_7 + 8\delta_9 + 4\delta_{10}$$

and

$$\nu^{*3} = 8\delta_3 + 24\delta_4 + 12\delta_5 + 8\delta_6 + 66\delta_7 + 54\delta_8 - \delta_9 + 54\delta_{10} + 66\delta_{11} + 8\delta_{12} + 12\delta_{13} + 24\delta_{14} + 8\delta_{15}.$$

We have

$$\nu^{*2} \geq 0, \quad \nu + \frac{1}{8}\nu^{*2} \geq 0, \quad \nu^{*2} + c\nu^{*3} \geq 0 \quad (0 \leq c \leq 1).$$

Hence

$$e(\nu) = \delta_0 + (\nu + \frac{1}{3}\nu^{*2}) + \frac{1}{6}(\nu^{*2} + \nu^{*3}) + \sum_{n=2}^{\infty} \frac{\nu^{*2(n-1)}}{(2n)!} * \Big(\nu^{*2} + \frac{\nu^{*3}}{2n+1}\Big).$$

Consequently, $e(\nu)$ is a finite non-negative measure on $\mathbb{Z}_+$ with $(e(\nu))(\mathbb{Z}_+) = e^{\nu(\mathbb{Z}_+)} = e^7$. Take $Z$ to be a random variable with distribution $e^{-7}e(\nu)$. Then the characteristic function of $Z$ equals $\phi$. Now let $X$ be a compound Poisson random variable, independent of $Z$, and with Lévy measure $\delta_3$. Then $X + Z$ is compound Poisson with Lévy measure $2\delta_1 + 2\delta_2 + 2\delta_4 + 2\delta_5$.

An interesting case where infinite divisibility of the skew factors is automatic occurs in the context of Mehler semigroups; we refer to [17] for the details.

## 3. Second Quantisation

Suppose $\lambda$ is an infinitely divisible Radon measure on a real Banach space $E$. Then we may write

$$\lambda = \gamma * \pi$$

with $\gamma$ a centred Gaussian Radon measure on $E$ and $\pi$ the distribution of a random variable of the form $\xi + \int_E x\, d\bar{\Pi}(x)$ as explained in the introduction.

For functions $f \in L^2(\lambda)$ put

$$F_f(x,y) := f(x+y), \qquad x, y \in E.$$

Using the fact that $L^2(\gamma)\widehat{\otimes}L^2(\pi) = L^2(\gamma \times \pi)$ isometrically (with $\widehat{\otimes}$ indicating the Hilbert space tensor product) it is immediate to verify that

$$\|f\|^2_{L^2(\lambda)} = \int_E \int_E |f(x+y)|^2\, d\gamma(x)\, d\pi(y) = \|F_f\|^2_{L^2(\gamma)\widehat{\otimes}L^2(\pi)}.$$

As a result the mapping $f \mapsto F_f$ is a linear isometry from $L^2(\lambda)$ into $L^2(\gamma)\widehat{\otimes}L^2(\pi)$. This brings us to the setting with independence structure as discussed in [2]. Following that reference, formally we define a derivative operator acting with dense domain in $L^2(\gamma) \otimes L^2(\pi)$ by the formula

$$D := D_\gamma \otimes I + I \otimes D_\pi,$$

where we denote the 'Gaussian' and the 'Poissonian' derivatives with subscripts $\gamma$ and $\pi$, respectively. Recall from [3] that these are defined as follows. The Gaussian derivative is defined by

$$D_\gamma f(x) := \sum_{n=1}^{N} \partial_n g(\phi_{h_1}(x), \dots, \phi_{h_N}(x)) \otimes h_n$$

for cylindrical functions $f = g(\phi_{h_1}, \dots, \phi_{h_N})$, with $g \in C^1_{\mathrm{b}}(\mathbb{R}^N)$ and $\phi : h \mapsto \phi_h$ being the isometry which embeds the reproducing kernel Hilbert space $H_\gamma$ of $\gamma$ onto the first Wiener-Itô chaos of $L^2(\gamma)$. The space of all such functions $f$ is dense in $L^2(\gamma)$ and $D_\gamma$ is closable as an operator from this initial domain into $L^2(\gamma; H_\gamma)$. The Poissonian derivative is defined by

$$D_\pi f(x) = f(x + \cdot) - f(x).$$

In order to prove that $D_\pi$ is densely defined as an operator from $L^2(\pi)$ into $L^2(\pi \times \nu)$ we need to find a dense set of functions $f$ in $L^2(\pi)$ such that $D_\pi f$ belongs to $L^2(\pi \times \nu)$. For this, we consider cylindrical functions $f$ of the form

$$f(x) = g(\langle x, x_1^* \rangle, \dots, \langle x, x_N^* \rangle)$$

with $g \in C^1_{\mathrm{b}}(\mathbb{R}^N)$ and $x_1^*, \dots, x_n^* \in E^*$. For such $f$ we have, where $0 < \theta_n(\cdot) < 1$ for each $n \in \mathbb{N}$,

$$\begin{aligned}
&\|D_\pi f\|^2 \\
&= \int_{E\times E} \Big| g(\langle x+y, x_1^*\rangle, \dots, \langle x+y, x_N^*\rangle) - g(\langle x, x_1^*\rangle, \dots, \langle x, x_N^*\rangle) \Big|^2\, d\pi(x)\, d\nu(y) \\
&= \int_{\{\|y\|>1\}\times E} \Big| g(\langle x+y, x_1^*\rangle, \dots, \langle x+y, x_N^*\rangle) - g(\langle x, x_1^*\rangle, \dots, \langle x, x_N^*\rangle) \Big|^2 d\pi(x)\, d\nu(y)
\end{aligned}$$

$$+ \int_{\{\|y\|\le 1\}\times E} \Big| \sum_{n=1}^N (\partial_n g(\langle x+\theta_1(x)y, x_1^*\rangle, \dots, \langle x+\theta_N(x)y, x_N^*\rangle))\langle y, x_n^*\rangle \Big|^2 \, d\pi(x)\, d\nu(y)$$

$$\le 4\|g\|_\infty^2 \nu\{\|y\|>1\} + \sum_{n=1}^N \|\partial_n g\|_\infty^2 \int_{\{\|y\|\le 1\}} |\langle y, x_n^*\rangle|^2 \, d\nu(y)$$

$$< \infty,$$

the finiteness in the last step being a consequence of the general properties of Lévy measures on Banach spaces (see [8, pp. 95–120] or [12, pp. 69–75]).

**Lemma 3.1.** *$D_\pi$ is closable as a densely defined linear operator from $L^2(\pi)$ to $L^2(\pi\times\nu)$.*

*Proof.* Suppose $f_n \to 0$ in $L^2(\pi)$ and $D_\pi f_n \to F$ in $L^2(\pi\times\nu)$. We must prove that $F = 0$. Passing to a subsequence, we may assume that $f_n(x) \to 0$ for $\pi$-almost all $x \in E$ and $D_\pi f_n(x,y) = f_n(x+y) - f_n(x) \to F(x,y)$ for $\pi\times\nu$-almost all $(x,y) \in E\times E$. Then, by Fubini's theorem, for $\nu$-almost all $y \in E$ we have $f_n(x+y) \to F(x,y)$ for $\pi$-almost all $x \in E$. Since for all $y \in E$ we have $f_n(x+y) \to 0$ for $\pi$-almost all $x \in E$, it follows that for $\nu$-almost all $y \in E$ we have $F(x,y) = 0$ for $\pi$-almost all $x \in E$. Using Fubini's theorem once more, it follows that $F(x,y) = 0$ for $\pi\times\nu$-almost all $(x,y) \in E\times E$. □

From now on, we use the notations $D_\gamma$ and $D_\pi$ for the closures of the operators considered so far and denote by $\mathsf{D}(D_\gamma)$ and $\mathsf{D}(D_\pi)$ their domains.

**Lemma 3.2.** *Suppose $T_1 : E_1 \to F_1$ and $T_2 : E_2 \to F_2$ are densely defined closed linear operators, with domains $\mathsf{D}(T_1)$ and $\mathsf{D}(T_2)$ respectively. Let $G$ be another Banach space and let $X\widehat{\otimes}Y$ denote the completion of $X\otimes Y$ with respect to any norm which has the property that $\|x\otimes y\| = \|x\|\|y\|$ for all $x\in X$ and $y \in Y$.*

1. *The operators $T_1\otimes I : E_1\widehat{\otimes}G \to F_1\widehat{\otimes}G$ and $I\otimes T_2 : G\widehat{\otimes}E_2 \to G\widehat{\otimes}F_2$ with their natural domains $\mathsf{D}(T_1)\otimes G$ and $G\otimes\mathsf{D}(T_2)$ are closable;*
2. *The operator $T_1\otimes I + I\otimes T_2 : E_1\widehat{\otimes}E_2 \to F_1\widehat{\otimes}F_2$ with its natural domain $\mathsf{D}(T_1)\otimes\mathsf{D}(T_2)$ is closable.*

*Proof.* Part (1) is immediate from the fact that $\|x\otimes y\| = \|x\|\|y\|$; part (2) follows from the fact that a densely defined linear operator is closable if and only if its domain is weak$^*$-densely defined, along with the operator inclusion $T_1^*\otimes I + I\otimes T_2^* \subseteq (T_1\otimes I + I\otimes T_2)^*$. The details are left to the reader. □

To proceed any further we define, for $n = 0, 1, 2, \dots$, the Hilbert spaces

$$\mathscr{H}_n := \bigoplus_{\substack{j,k\ge 0\\ j+k=n}} H_\gamma^{\circledS j}\widehat{\otimes}L^2(\nu)^{\circledS k}.$$

We use the convention that $G^{\circledS 0} = \mathbb{R}$ for any Hilbert space $G$ and recall that $\widehat{\otimes}$ refers to the Hilbertian completion of the algebraic tensor product. We set

$$\mathscr{H} := \mathscr{H}_1 = (H_\gamma\widehat{\otimes}\mathbb{R}) \oplus (\mathbb{R}\widehat{\otimes}L^2(\nu)) = H_\gamma \oplus L^2(\nu).$$

Having defined $D_\gamma$ (respectively $D_\pi$) as closed densely defined operators from $L^2(\gamma)$ into $L^2(\gamma)\widehat{\otimes}H_\gamma$ (respectively from $L^2(\pi)$ into $L^2(\pi\times\nu) = L^2(\pi)\widehat{\otimes}L^2(\nu)$), we now identify both $L^2(\gamma)\widehat{\otimes}H_\gamma$ and $L^2(\pi)\widehat{\otimes}L^2(\nu)$ canonically with closed subspaces of $(L^2(\gamma)\widehat{\otimes}L^2(\pi))\widehat{\otimes}(H_\gamma\oplus L^2(\nu)) = L^2(\gamma\times\pi;\mathscr{H})$. We denote by $D_\gamma\otimes I$ and $I\otimes D_\pi$

the resulting closed and densely defined operators from $L^2(\gamma)\widehat{\otimes}L^2(\pi) = L^2(\gamma\times\pi)$ into $L^2(\gamma\times\pi;\mathscr{H})$, and define

$$D = D_\gamma \otimes I + I \otimes D_\pi.$$

By part (1) of the previous lemma, after completing we can consider $D_\gamma\otimes I$ and $I\otimes D_\pi$ as closed and densely defined operators from $L^2(\gamma\times\pi;\mathscr{H}_n)$ into $L^2(\gamma\times\pi;\mathscr{H}_{n+1})$,

By combining the preceding two lemmas we obtain the following result.

**Proposition 3.3.** *For all $n = 0,1,2,\dots,$ the operator $D = D_\gamma\otimes I + I\otimes D_\pi$ is closable as a densely defined operator from $L^2(\gamma\times\pi;\mathscr{H}_n)$ into $L^2(\gamma\times\pi;\mathscr{H}_{n+1})$.*

We define the $n$-fold stochastic integral on $I_n : \mathscr{H}_n \to L^2(\Omega)$ by

$$I_n(f\otimes g) := I_{j,\gamma} f \otimes I_{k,\pi} g$$

for $f\in H_\gamma^{\circledS j}$ and $g\in L^2(\nu)^{\circledS k}$ with $j+k=n$, where we denote the 'Gaussian' and the 'Poissonian' integrals with subscripts $\gamma$ and $\pi$, respectively.

In what follows, in order to tidy up the notation we will refrain from writing subscripts $\gamma$ and $\pi$; expectations taken in the the left and right sides of tensor products refer to $\gamma$ and $\pi$, respectively.

Let $\Pi$ be a Poisson random measure on a probability space $(\Omega,\mathscr{F},\mathbb{P})$, whose intensity measure $\nu$ is a Lévy measure on $E$. Recall that the former means that $\Pi$ is a random variable on $(\Omega,\mathscr{F},\mathbb{P})$ taking values in the space $\mathbb{N}(E)$ of $\mathbb{N}$-valued measures on $E$ endowed with the $\sigma$-algebra generated by the Borel sets of $E$, that is, the smallest $\sigma$-algebra which renders the mappings $\xi\mapsto\xi(B)$ measurable for all $B\in\mathscr{B}(E)$. By $\mathbb{P}_\Pi$ we denote the image measure of $\mathbb{P}$ under $\Pi$.

Following Last and Penrose [11], for a measurable function $f:\mathbb{N}(Y)\to\mathbb{R}$ and $y\in Y$ we define the measurable function $\tilde{D}_y f:\mathbb{N}(Y)\to\mathbb{R}$ by

$$\tilde{D}_y f(\eta) := f(\eta+\delta_y) - f(\eta).$$

The function $\tilde{D}^n_{y_1,\dots,y_n} f:\mathbb{N}(Y)\to\mathbb{R}$ is defined recursively by

$$\tilde{D}^n_{y_1,\dots,y_n} f = \tilde{D}_{y_n}\tilde{D}^{n-1}_{y_1,\dots,y_{n-1}} f,$$

for $y_1,\dots,y_n\in Y$. This function is symmetric, i.e. it is invariant under any permutation of the variables.

Following [3], we define $j : L^2(E,\mu)\to L^2(\mathbb{P}_\Pi)$ by

$$jf(\eta) = f\Big(\xi + \int_E x\,\bar{\eta}(dx)\Big), \quad \eta\in\mathbb{N}(E).$$

The rigorous interpretation of this identity is provided by noting that

$$\|jf\|^2_{L^2(\mathbb{P}_\Pi)} = \mathbb{E}\Big|f\Big(\xi+\int_E x\,d\bar{\Pi}(x)\Big)\Big|^2 = \|f\|^2_{L^2(E,\mu)},$$

which means that $jf(\eta)$ is well-defined for $\mathbb{P}_\Pi$-almost all $\eta$ and that $j$ establishes an isometry from $L^2(E,\mu)$ into $L^2(\mathbb{P}_\Pi)$. Note that

$$jf(\Pi) = f\Big(\xi + \int_E x\,d\bar{\Pi}(x)\Big)$$

and

$$j\circ D = \tilde{D}\circ j.$$

We now have the following extension to infinitely divisible measures of the corresponding results of Stroock [18] (for Gaussian measures) and Last and Penrose [11] (for Poisson random measures):

**Proposition 3.4.** *For all $f \in W^{\infty,2}(\gamma)$ and $g \in L^2(\mathbb{P}_\Pi)$ we have*

$$f \otimes g(\Pi) = \sum_{m=0}^{\infty} \frac{1}{m!} I_m(\mathbb{E}(\tilde{D}^m f \otimes g(\Pi))).$$

*Proof.* By Leibniz's rule,

$$\begin{aligned}\sum_{m=0}^{\infty} \frac{1}{m!} I_m \mathbb{E}_\lambda \tilde{D}^m f \otimes g(\Pi) &= \sum_{m=0}^{\infty} \frac{1}{m!} I_m\Big(\mathbb{E}_\lambda \sum_{\ell=0}^{m} \binom{m}{\ell} \tilde{D}^\ell f \otimes \tilde{D}^{m-\ell} g(\Pi)\Big)\\ &= \sum_{m=0}^{\infty}\sum_{\ell=0}^{m} \frac{1}{\ell!(m-\ell)!} I_m\Big(\mathbb{E}_\lambda\big(\tilde{D}^\ell f \otimes \tilde{D}^{m-\ell} g(\Pi)\big)\Big)\\ &= \sum_{m=0}^{\infty}\sum_{\ell=0}^{m} \frac{1}{\ell!(m-\ell)!} I_\ell(\mathbb{E}_\lambda \tilde{D}^\ell f) \otimes I_{m-\ell}(\mathbb{E}_\pi \tilde{D}^{m-\ell} g(\Pi))\\ &= \sum_{j=0}^{\infty} \frac{1}{j!} I_j(\mathbb{E}_\gamma \tilde{D}^j f) \otimes \sum_{k=0}^{\infty} \frac{1}{k!} I_k(\mathbb{E}_\pi \tilde{D}^k g(\Pi))\\ &= f \otimes g(\Pi).\end{aligned}$$

using the Stroock and Last-Penrose type decompositions in the penultimate identity. □

We now return to the setting considered in Section 2 and make the standing assumption that the equivalent conditions stated in Proposition 2.1 are satisfied. Thus we assume that $\lambda_1 = \gamma_1 * \pi_1$ on $E_1$, $\lambda_2 = \gamma_2 * \pi_2$ on $E_2$, and that $T : E_1 \to E_2$ is a Borel linear skew mapping with respect to the pair $(\lambda_1, \lambda_2)$ with an infinite divisible skew factor. As is shown by Proposition 2.1, this implies that $T$ is skew with respect to both pairs $(\gamma_1, \gamma_2)$ and $(\pi_1, \pi_2)$, that is, $T\gamma_1 * \rho_\gamma = \gamma_2$ and $T\pi_1 * \rho_\pi = \pi_2$.

It follows Proposition 2.1 that we may define $P_T : L^2(E_2, \lambda_2) \to L^2(E_1, \lambda_1)$ by

$$P_T f(x) := \int_{E_2} f(Tx + y)\, d\rho(y), \quad x \in E_1,$$

where $\rho := \rho_\gamma * \rho_\pi$ is the skew factor on $E_2$, i.e., $T\lambda_1 * \rho = \lambda_2$. Similarly we can define an operator $P_T \otimes P_T : L^2(\gamma_2) \otimes L^2(\pi_2) \to L^2(\gamma_1) \otimes L^2(\pi_1)$ in the obvious way (with slight abuse of notation; we should really be writing $P_{\gamma,T} \otimes P_{\pi,T}$) and we then have:

**Lemma 3.5.** *Under the above assumptions, $F_{P_T f} = (P_T \otimes P_T) F_f$.*

*Proof.* For $(\gamma \times \pi)$-almost all $x, y \in E_2$ we have

$$\begin{aligned}(P_T \otimes P_T)(\phi \otimes \psi)(x, y) &= (P_T\phi \otimes P_T\psi)(x, y)\\ &= \int_{E_2} \phi(Tx + z)\, d\rho_\gamma(z) \int_{E_2} \psi(Ty + z)\, d\rho_\pi(z)\\ &= \int_{E_2}\int_{E_2} (\phi \otimes \psi)(Tx + z_1, Ty + z_2)\, d\rho_\gamma(z_1)\, d\rho_\pi(z_2).\end{aligned}$$

Now suppose that $F_f = \lim_{n\to\infty} G_n$ in $L^2(\gamma\times\pi)$, where each $G_n$ belongs to the algebraic tensor product $L^2(\gamma)\otimes L^2(\pi)$. By the above identity and linearity it follows, after passing to a subsequence if necessary, that for $(\gamma\times\pi)$-almost all $x, y \in E_2$ we have

$$\begin{aligned}
(P_T\otimes P_T)F_f(x,y) &= \lim_{n\to\infty}(P_T\otimes P_T)G_n(x,y)\\
&= \lim_{n\to\infty}\int_{E_2}\int_{E_2} G_n(Tx+z_1,Ty+z_2)\,d\rho_\gamma(z_1)\,d\rho_\pi(z_2)\\
&= \int_{E_2}\int_{E_2} F_f(Tx+z_1,Ty+z_2)\,d\rho_\gamma(z_1)\,d\rho_\pi(z_2)\\
&= \int_{E_2}\int_{E_2} f(Tx+Ty+z_1+z_2)\,d\rho_\gamma(z_1)\,d\rho_\pi(z_2)\\
&= \int_{E_2} f(Tx+Ty+z)\,d(\rho_\gamma*\rho_\pi)(z)\\
&= \int_{E_2} f(Tx+Ty+z)\,d\rho(z)\\
&= P_Tf(x+y)\\
&= F_{P_T}(x,y).
\end{aligned}$$

□

For $h\in H_\gamma$ and $y_1,\dots,y_n\in E$ and $h\in H_\gamma$ we define

$$D_{h;y} := D_h\otimes I + I\otimes D_y,$$

where

$$D_hf(x) := \langle D_\gamma f(x), h\rangle, \quad D_yg(x) := (D_\pi g(x))(y).$$

For the higher order derivatives we define inductively

$$D^n_{h_1,\dots,h_n;y_1,\dots,y_n} := D_{h_n;y_n}D^{n-1}_{h_1,\dots,h_{n-1};y_1,\dots,y_{n-1}}.$$

**Lemma 3.6.** *For all $f\in L^2(E_2,\lambda_2)$, $h\in H_\gamma$, and $y_1,\dots,y_n\in E_1$,*

$$\mathbb{E}_{\gamma_1\times\pi_1}D^n_{h_1,\dots,h_n;y_1,\dots,y_n}F_{P_Tf} = \mathbb{E}_{\gamma_2\times\pi_2}D^n_{Th_1\dots,Th_n;Ty_1,\dots,Ty_n}F_f. \tag{3.1}$$

*Proof.* We approximate $F_f$ by finite sums of elementary tensors as in the proof of the previous lemma. For such functions $G_n$ the identity follows from the results in [3] for the Gaussian and Poissonian case. Thanks to the closedness of the derivative operators, the identity passes over to the limit. □

For Hilbert spaces $H$ and $\underline{H}$ we note that

$$\Gamma(H\oplus\underline{H}) = \bigoplus_{n=0}^{\infty}\Big(\bigoplus_{\substack{j,k\ge0\\ j+k=n}} H^{\circledS j}\widehat{\otimes}\underline{H}^{\circledS k}\Big).$$

Putting everything together we obtain the following result which generalises the results of Theorems 3.5 and 4.4 of [3], where Gaussian and Poisson noises were treated separately.

**Theorem 3.7.** *Under the standing assumption stated above, the following diagram commutes:*

$$\begin{array}{ccc}
L^2(E_2,\lambda_2) & \xrightarrow{P_T} & L^2(E_1,\lambda_1) \\
{\scriptstyle f\mapsto F_f}\big\downarrow & & \big\downarrow{\scriptstyle f\mapsto F_f} \\
L^2(E_2\times E_2,\gamma_2\times\pi_2) & \xrightarrow{P_T\otimes P_T} & L^2(E_1\times E_1,\gamma_1\times\pi_1) \\
{\scriptstyle \bigoplus_{n=0}^\infty \frac{1}{\sqrt{n!}}\mathbb{E}_{\gamma_2\times\pi_2}D^n}\big\downarrow & & \big\downarrow{\scriptstyle \bigoplus_{n=0}^\infty \frac{1}{\sqrt{n!}}\mathbb{E}_{\gamma_1\times\pi_1}D^n} \\
\Gamma((H_{\gamma,2}\oplus L^2(E_2,\nu_2)) & \xrightarrow{\bigoplus_{n=0}^\infty (T^*)^{\circledS n}} & \Gamma(H_{\gamma,1}\oplus L^2(E_1,\nu_1))
\end{array}$$

*Moreover, for $k = 1, 2$ also the following diagram commutes in distribution if $X_k$ is an $E$-valued random variable with distribution $\lambda_k$:*

$$\begin{array}{ccc}
L^2(E_k\times E_k,\gamma_k\times\pi_k) & \xrightarrow{(f,g)\mapsto(f(X_{\gamma,k}),f(X_{\pi,k}))} & L^2(\Omega\times\Omega) \\
{\scriptstyle \bigoplus_{n=0}^\infty \frac{1}{\sqrt{n!}}\mathbb{E}_{\gamma_k\times\pi_k}D^n}\big\downarrow & & \big\uparrow{\scriptstyle \bigoplus_{n=0}^\infty \frac{1}{\sqrt{n!}}I_n} \\
\Gamma(H\oplus L^2(E_k,\nu_k)) & \xrightarrow{=} & \Gamma(H\oplus L^2(E_k,\nu_k))
\end{array}$$

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