Review of R.L.Schilling, R.Song, Z.Vondraček "Bernstein Functions" (second edition). De Gruyter Studies in Mathematics **37**, *(2012)*

Bernstein functions (and the related notion of completely monotone functions) play important roles in a variety of areas of mathematics, most notably real, complex and functional analysis and probability theory. This impressive monograph, which is now in an expanded second edition, does a wonderful job of collecting together and synthesising some of the most important results and applications. Consequently it represents a unique perspective on this important function class which will be a valuable resource for many years to come for both experienced researchers and graduate students. I will now describe the contents of the book chapter-by-chapter.

A C^∞ mapping $f : (0, \infty) \to \mathbb{R}$ is *completely monotone* (CM) if $(-1)^n f^{(n)} \geq 0$ for all $n \in \mathbb{N} \cup \{0\}$. Chapter 1 gives a brief account of such mappings and includes the important structural theorem that a function is CM if and only if it is the Laplace transform of a finite measure. Chapter 2 is about the subclass of CMs called *Stieltjes functions* (SF) which are essentially double Laplace transforms of suitable measures. In Chapter 3 we meet the epoynymous *Bernstein functions* (BF) which are C^∞ mappings $f : (0, \infty) \to [0, \infty)$ such that $(-1)^{n-1} f^{(n)} \geq 0$ for all $n \in \mathbb{N}$ and establish their canonical representations. An important structural result is that f is BF if and only if e^{-tf} is CM for all $t > 0$. Chapter 5 relates these ideas to the important classes of positive definite and negative definite functions, indeed the class of bounded CM functions is precisely the bounded continuous positive definite functions on $[0, \infty)$ and the BFs are the continuous negative definite functions on $[0, \infty)$. Chapter 5 then gives the probabilistic interpretation of these concepts so if $S = (S(t), t \geq 0)$ is a *subordinator* , i.e. an increasing Lévy process, then its Laplace exponent is a BF f, i.e. $\mathbb{E}(e^{-uS(t)}) = e^{-tf(u)}$ for all $t, u > 0$ and the canonical representation of f is the *Lévy-Khinchine formula* for the process S.

In Chapter 6 we meet *complete Bernstein functions* (CBF) which are precisely those BF for which the Lévy measure has a CM density. Here there is an important connection to complex analysis, in that every CBF has an analytic extension to, and which preserves, the complex upper half-plane. In fact these extensions are precisely those *Pick-Nevanlinna* functions whose restrictions to $(0, \infty)$ are non-negative. The structure of CBFs is further developed in Chapter 7 where we learn that f is a CBF if and only if $1/f$ is a SF. In Chapter 8 we meet the class of *Thorin-Bernstein functions* which are those BF whose Lévy measure has a density m so that $t \to tm(t)$ is CM. In Chapter 9 we return to probability theory and introduce the *Bondesson* and *generalised gamma* distributions which are classes of infinitely divisible sub-probability measures on $(0,\infty)$ that are generated by taking vague limits of convolutions of mixtures of exponential distributions and gamma distributions (respectively.) These are characterised as having Laplace exponents which are CBF and TBF (respectively.) The probabilistic theme continues in Chapter 10 where transforms of BFs are studied which relate to the distribution of the random variable $X = \int_{(0,T)} \theta(t)dS(t)$ where the mapping θ is strictly decreasing. These distributions include variants on the important class of *self-decomposable* ones. In Chapter 11 we meet *special Bernstein functions* SBF which are those BF f such that f^* is also a BF where $f^*(s) = s/f(s)$. This class includes CBF and is of interest because it corresponds to those subordinators whose potential measure (restricted to $(0,\infty)$) has a non-increasing density. A sufficient condition for a non-increasing function to be the potential density of a subordinator is that it be log-convex and the Laplace transforms of such functions form the *Hirsch class* of CM functions which is studied in some detail in the second part of the chapter.

Applications to functional analysis begin in Chapter 12. If T is a dissipative self-adjoint operator on a complex Hilbert space H having resolvent $R_\lambda = (\lambda - T)^{-1}$ then for each $u \in H$ the mapping $f(\lambda) = \lambda\langle R_\lambda u, u\rangle$ is the extension of a CBF. This fact leads to a proof of the spectral theorem for such operators. In the second part of the chapter (positive) operator monotone functions are introduced and shown to be in one-to-one correspondence with CBF. In Chapter 13, the authors study the subordination of one-parameter semigroups of linear operators in Banach space and present Phillip's theorem for the representation of the generators of the subordinated semigroup and the associated functional calculus. It is shown that various functional inequalities for semigroups (Nash, Sobolev and weak/super Poincaré) are preserved by subordination and conditions are found on a BF for a subordinated ultracontractive semigroup to be either hypercontractive, supercontractive or ultracontractive. Finally the authors obtain some estimates for eigenvalues corresponding to the subordination of a symmetric Hunt process that is killed on exit from a region in $\mathbb{R}^d$.

In Chapter 14 we return to the potential theory of Brownian motion killed on exit from a region and having intrinsically ultracontractive transition semigroup. If f is a SBF then subordination of the process using f and f^* yields useful information about the excessive functions of the killed Brownian motion. In Chapter 15 the authors study generalised diffusions and show that these are in one-to-one correspondence with CBFs which are obtained as the Laplace exponents of the inverse local time of the process at zero. Finally Chapter 16 presents a very useful tabulated list of CBFs.