Extending Stochastic Resonance for Neuron Models to General Levy Noise

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Abstract

A recent paper by Patel and Kosko [5] demonstrated stochastic resonance for general feedback continuous and spiking neuron models using additive Levy noise constrained to have finite second moments. In this short paper we drop this constraint and show that their result extends to general Levy noise models. We achieve this by showing that "large jump" discontinuities in the noise can be controlled so as to allow the stochastic model to tend to a deterministic one as the noise dissipates to zero. Stochastic resonance then follows by a "forbidden intervals" theorem as in [5].

Index terms: Levy noise, neuron models, stochastic resonance, stochastic differential equation.

Stochastic resonance (SR) is a phenomenon wherein small amounts of random noise can enhance the output of a system rather than degrading it (see e.g. [2]). This has found a wide range of applications in physics, biology and medicine (see e.g. [3] and the extensive bibliography in [5]). However almost all applications up to now have employed Gaussian noise which has continuous sample paths. On the other hand Levy processes form a rich class of stochastic processes whose paths may contain random jump discontinuities of arbitrary size occurring at arbitrary random times and these are now being applied in many different areas such as financial economics and quantum physics (see [1] and references therein.)

Patel and Kosko [5] have recently published the first paper dealing with SR where the noise is a general Levy process, however they restricted to the case where the noise has finite variance. The aim of this paper is to demonstrate stochastic resonance (SR) in continuous and spiking neuron models using arbitrary driving Levy processes. A Levy process is essentially a stochastic process with stationary and independent increments. Examples are Brownian motion, the Poisson process and also non-Gaussian α-stable processes ($0 < \alpha < 2$) which have infinite variance (and also infinite mean if $\alpha \leq 1$) and self-similar sample paths. In [5] the authors show that Levy noise can lead to SR in noisy feedback neuron models where the noise enters additively, but they required the assumption that the Levy noise has a finite second moment. This excludes many important examples, such as the α-stable processes mentioned above, where simulation indicates that SR will also occur. The purpose of this note is to show that the finite second moment assumption can be dropped and so to establish SR for arbitrary driving Levy processes.

We use the same notation and set-up as in [5] so our driving noise is a Levy process $L_t = (L_t^1, L_t^2, \ldots, L_t^m)$ taking values in $\mathbb{R}^m$ that is defined on a probability space $(\Omega, \mathcal{F}, P)$ which is equipped with a filtration $(\mathcal{F}_t, t \geq 0)$. As in [5] we make the convenient assumption that each $L_t^j (1 \leq j \leq m)$ is a one-dimensional Levy process and that these component processes are independent. We employ the Levy-Ito decomposition (see e.g. [1]) to decompose the component process L_t^j into continuous and jump parts:

$$L_t^j = \mu^j t + \sigma^j B_t^j + \int_{|y^j|<1} y^j \tilde{N}^j(t, dy^j) + \int_{|y^j|\geq 1} y^j N^j(t, dy^j). \tag{1}$$

where for each $1 \leq j \leq m, \mu^j \in \mathbb{R}, \sigma^j \geq 0, (B_t^j, t \geq 0)$ is a standard Brownian motion (Bm) and N^j is a Poisson random measure defined on $\mathbb{R}^+ \times (\mathbb{R} - \{0\})$ which is independent of the Bm and has intensity measure $dt\nu^j(dy^j)$ where ν^j is a Levy measure. The compensated random measure is $\tilde{N}^j(dt, dy^j) = N^j(t, dy^j) - dt\nu^j(dy^j)$. For each $1 \leq j \leq m$ define $P_t^j = \int_{|y^j|\geq 1} y^j N^j(t, dy^j)$ and $M_t^j = L_t^j - P_t^j$. Then $(M_t^j, t \geq 0)$ and $(P_t^j, t \geq 0)$ are independent Levy processes where the jump sizes of the process M_t^j are all bounded by one. It follows from Theorem 2.4.7 in [1] that M_t^j has finite moments to all orders.

To describe continuous neuron models with additive Levy noise Patel and Kosko [5] introduce the stochastic differential equation (SDE)

$$dX_t = b(X_{t-})dt + c(X_{t-})dL_t, \tag{2}$$

where $X_t = (X_t^1, \ldots, X_t^d)$, b^i and c_j^i are globally Lipschitz functions and we have the global bound

$$\sup_{x \in \mathbb{R}^d} |c_j^i(x)|^2 \leq H_j^i. \tag{3}$$

In order to focus on "pure noise" effects we take $\mu^j = 0$ as in [5]. There is no loss of generality here as μ^j can always be incorporated into the drift term b. Now consider the noiseless version of (2):

$$d\widehat{X}_t = b(\widehat{X}_{t-})dt. \tag{4}$$

A key step on the way to obtaining SR in ([5]) is Lemma 1 therein where it is shown that the solution to (2) converges to that of (4) in probability as the noise dissipates to zero. Specifically it is shown that (under the square-integrability assumption) for all $T > 0, K > 0$:

$$P\left(\sup_{0 \leq t \leq T} ||X_t - \widehat{X}_t|| > K\right) \to 0 \tag{5}$$

as $\sigma^j \to 0$ and $\nu^j \to 0$ for all $1 \leq j \leq m$. The remainder of this note is concerned with the extension of (5) to general Levy noise. Specifically we have the following:

Theorem 1 *For each $1 \leq j \leq d$, let L_t^j be an arbitrary real-valued Lévy process (so it has the form (1)) and assume that the L_t^js are independent stochastic processes. Then for all $T > 0, K > 0$:*

$$P\left(\sup_{0 \leq t \leq T} ||X_t - \widehat{X}_t|| > K\right) \to 0$$

as $\sigma^j \to 0$ and $\nu^j \to 0$ for all $1 \leq j \leq m$.

Proof. We first rewrite (2) as

$$dX_t = b(X_{t-})dt + c(X_{t-})dM_t + c(X_{t-})dP_t. \tag{6}$$

For each $1 \leq i \leq d$ and $t \geq 0$, define

$$Z_i(t) = \int_0^t c_i^j(X_{s-})dP_s^j = \int_0^t \int_{|y^j| \geq 1} c_i^j(X_{s-})y^j N^j(ds, dy^j)$$

and write $Z(t) = (Z_1(t), \ldots, Z_d(t))$. By (5) we have

$$P\left(\sup_{0 \leq t \leq T} ||X_t - \widehat{X}_t - Z(t)|| > K\right) \to 0, \tag{7}$$

as $\sigma^j \to 0$ and $\nu^j \to 0$ for all $1 \leq j \leq m$, so in order to establish the required result we need only show that

$$P\left(\sup_{0\leq t\leq T} ||Z(t)|| > K\right) \to 0, \tag{8}$$

as $\nu^j \to 0$ for all $1 \leq j \leq m$ where we here defining $\nu_j = \nu_j(A)$ where $A = (-\infty, -1] \cup [1, \infty)$. Using the Cauchy-Schwarz inequality for sums we have

$$\begin{aligned} P\left(\sup_{0\leq t\leq T} ||Z(t)|| > K\right) &= P\left(\sup_{0\leq t\leq T} ||Z(t)||^2 > K^2\right) \\ &\leq P\left(\sum_{1\leq i\leq d} \sup_{0\leq t\leq T} |Z_i(t)|^2 > K^2\right) \\ &\leq P\left(\max_{1\leq i\leq d} \sup_{0\leq t\leq T} |Z_i(t)|^2 > \frac{K^2}{d}\right) \\ &= P\left(\max_{1\leq i\leq d} \sup_{0\leq t\leq T} |Z_i(t)| > \frac{K}{\sqrt{d}}\right), \end{aligned}$$

and so our goal is reached if we can prove that for all $1 \leq i \leq d, T \geq 0, K > 0$:

$$P\left(\sup_{0\leq t\leq T} |Z_i(t)| > K\right) \to 0 \text{ as } \max_{1\leq j\leq m} \nu_j \to 0. \tag{9}$$

Define $h_i = \max_{1\leq j\leq m} \sqrt{H_i^j}$ then by (3),

$$|Z_i(t)| \leq \max_{1\leq j\leq m} \sup_{0\leq s\leq t} |c_i^j(X_{s-})| \sum_{j=1}^m Q_j(t) \leq h_i \sum_{j=1}^m Q_j(t),$$

where $Q_j(t) = \int_{|y^j|\geq 1} |y^j| N^j(t, dy^j)$, for $1 \leq j \leq m, t \geq 0$. We use the elementary inequality $P(Y > K) \geq P(X > K)$ for random variables $Y \geq X \geq 0$ to see that

$$P\left(\sup_{0\leq t\leq T} |Z_i(t)| > K\right) \leq P\left(\sum_{j=1}^m \sup_{0\leq t\leq T} Q_j(t) > \frac{K}{h_i}\right) \leq \sum_{j=1}^m P\left(\sup_{0\leq t\leq T} Q_j(t) > \frac{K}{mh_i}\right),$$

where the second inequality follows from the fact that for random variables $X_1, \ldots, X_m, P(|X_1 + \cdots X_m| > K) \leq \sum_{j=1}^m P\left(|X_j| > \frac{K}{m}\right)$.

Hence to establish (9) it is sufficient to prove that for each $1 \leq j \leq d, L > 0$,

$$P\left(\sup_{0 \leq t \leq T} Q_j(t) > L\right) \to 0 \text{ as } \nu_j \to 0. \tag{10}$$

It is shown in Chapter 2 of [1] that $Q_j = (Q_j(t), t \geq 0)$ is a compound Poisson process and that we can write $Q_j(t) = \sum_{n=1}^{N_j(t)} W_{j,n}$ where $(W_{j,n}, n \in \mathbb{N})$ is a sequence of non-negative i.i.d. random variables having common law

$$p_{W_j}(B) = \frac{\nu_j(-B \cap [(-\infty, -1])) + \nu_j(B \cap [1, \infty))}{\nu_j}$$

and $(N_j(t), t \geq 0)$ is an independent Poisson process having intensity ν_j. It follows that $\sup_{0 \leq t \leq T} Q_j(t) = Q_j(T)$ since $(N_j(t), t \geq 0)$ is non-decreasing.

The probability law of $Q_j(T)$ is

$$\begin{aligned} \mu_j(B) &= \sum_{n=0}^{\infty} e^{-T\nu_j} \frac{T^n \nu_j^n}{n!} p_{W_j}^{*n}(B) \\ &= \sum_{n=0}^{\infty} e^{-T\nu_j} \frac{T^n}{n!} \widetilde{\nu_j}^{*n}(B), \end{aligned}$$

(see e.g. Chapter VI, section 4 in [4]) where $*n$ denotes the nth convolution power and $\widetilde{\nu_j}(B) = \nu_j(-B \cap [(-\infty, -1])) + \nu_j(B \cap [1, \infty))$.

It is easy to see that for all $n \in \mathbb{N}, \widetilde{\nu_j}^{*n}(B) \to 0$ as $\nu_j \to 0$ and so by dominated convergence it follows that $\mu_j(B) \to 0$ as $\nu_j \to 0$. We obtain (10) when we take $B = (L, \infty)$. □

SR follows from the result of Theorem 1 by the argument of Theorem 1 in [5]. The same arguments allow us to extend Lemma 2 of [5] to general Levy noise and hence obtain SR for spiking neuron models. We remark that the condition that the L_t^js are independent Levy processes, which is built into the model in [5], can be dropped and the results of this paper then extend easily to the case where L_t is an arbitrary $\mathbb{R}^m$-valued Levy process.

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