Review of Stroock "Mathematics of Probability" (2013)

This book is a very thorough advanced undergraduate/beginning graduate course on probability theory for students who have a good background in modern mathematical ideas. To be more specific, the typical reader should know a little topology, and not be frightened by lengthy analytical arguments. Measure theory is not assumed – it is developed from scratch by the author. The main subject matter is in a sense quite standard, and what distinguishes this book from its many competitors is the thoroughness of argument, and the tasteful choice of auxiliary topics that complement the main menu. In the chapter-by-chapter account that follows, I will emphasise some of highlights from the latter.

Chapter 1 gives some background in finite and countable sample spaces. Examples considered include random walks and the application of random graphs to study tournaments. In particular, we meet Erdös' approach to finding a sufficient condition for a tournament to have a given property. Random walks are an appropriate context to study de Moivre's central limit theorem the arcsine law, recurrence and transience, conditioning and independence. In Chapter 2 we move onto uncountable sample spaces, and this motivates the author to give a comprehensive account of a great deal of what the probabilist-in-the-street needs to know about measure theory and Lebesgue integration. I appreciated a slick proof of Jensen's inequality and the derivation of Stirling's formula for the gamma function.

Chapter 3 applies the Lebesgue theory to give a thorough treatment of independence and conditioning. Kolmogorov's inequality for sums of independent random variables is proved and used to establish the strong law of large numbers. Some readers may mourn the absence of Fourier analytic methods, about which the author comments (p.134) "they often work so well that they can mask probabilistic insight" (sic). Gaussian random variables and vectors are the subject matter for the short Chapter 4. Here we find proofs of the central limit theorem and of Lindeberg's theorem. It is pleasing to find readers also being exposed to the beautiful topic of Gaussian concentration inequalities via the Maury-Pisier estimate.

In Chapter 5 we begin the study of discrete-time stochastic processes. First we return to random walks and investigate the gambler's ruin problem. Then we turn to Markov chains. The variational distance between measures is introduced, and Doeblin's theorem proved which gives a sufficient condition for a Markov operator to be a contraction in this metric, and also for the corresponding Markov chain to have a unique invariant measure. Finally the ergodic theorem for stationary Markov chains is established. In Chapter 6 we move on to continuous time Markov chains. Readers meet the

Chapman-Kolmogorov equations, and the relatively simple example of the Poisson process is developed. Then we move on to Brownian motion, which is constructed using Paul Lévy's method of discrete approximants. Standard properties of Brownian motion are developed and the chapter closes with a construction of the Ornstein-Uhlenbeck process.

The final Chapter 7 is all about martingales, mainly in discrete time. Highlights include the use of martingale techniques to derive the Hardy-Littlewood maximal inequality, and hence Lebesgue's differentiation theorem for measures. The martingale convergence theorem is established and applied to prove the Radon-Nikodym theorem. Stopping times are introduced, and reversed martingales are used to establish the de Finetti exchangeability theorem, and hence the Hewitt-Savage $0 - 1$ law. The last part of the chapter presents a brief survey of continuous time martingales which should inspire students to learn Itô calculus, and concludes with the derivation of a formula of Feynman-Kac type.

The book is replete with carefully chosen exercise for readers to test their understanding. Another nice touch is that the author always takes care to let the reader know who originally came up with a particularly clever argument or method. In this way, readers get a healthy exposure to ways of thinking originating from Doeblin, Doob, Dynkin, Huygens, Kac, Kolmogorov, Lévy, Marcinkiewicz and Wiener, among many others. This is a very good book on which to base a graduate course or to use for self-study.