NONLINEAR MARKOV PROCESSES AND KINETIC EQUATIONS
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This monograph analyses a class of continuous time Markov processes that are of a suitable form to model complex dynamical systems such as coalescence, mass-exchange and fragmentation of interacting particle systems. The random dynamics that is described by these processes is of a canonical form that has attracted much attention in probability theory in that it can be decomposed as a superposition of three terms which describe deterministic evolution, diffusion and jump discontinuities (respectively.)

First some preliminaries. A Markov process is essentially a stochastic process for which the "past" and "future" are independent, given the "present". To be more specific, let $(\Omega, \mathcal{F}, P)$ be a probability space that is equipped with a filtration $(\mathcal{F}_t, t \geq 0)$ of sub-σ-algebras of $\mathcal{F}$ so that $\mathcal{F}_s \subseteq \mathcal{F}_t$ when $s \leq t$. A stochastic process $(X(t), t \geq 0)$ which takes values in a locally compact space E is a *Markov process* if each random variable $X(t)$ is $\mathcal{F}_t$-measurable and for every $0 \leq s \leq t < \infty$ and all bounded real-valued Borel measurable functions f on E:

$$\mathbb{E}(f(X(t))|\mathcal{F}_s) = \mathbb{E}(f(X(t))|X(s)) \text{ almost surely.}$$

We can view the process from a purely analytic perspective by "averaging" and considering the linear operators $T_{s,t}$ defined by $T_{s,t}f(x) = \mathbb{E}(f(X(t))|X(s) = x)$. These form a two-parameter evolution family in that $T_{r,s} \circ T_{s,t} = T_{r,t}$ whenever $r \leq s \leq t$. From now on we only consider the case where the process is time-homogeneous so that $T_{t-s} := T_{0,t-s} = T_{s,t}$. Let $C_0(E)$ be the Banach space of continuous functions on E which vanish at infinity equipped with the usual supremum norm. We say that $(X(t), t \geq 0)$ is a *Feller process* if for each $t \geq 0$

$$T_t : C_0(E) \to C_0(E) \text{ and } \lim_{t \to 0} ||T_t f - f|| = 0,$$

for all $f \in C_0(E)$. Then $(T_t, t \geq 0)$ is a C_0-semigroup on the space $C_0(E)$. In particular it has a closed, densely defined infinitesimal generator $\mathcal{A}$ and valuable information about the dynamics may be contained within the structure of $\mathcal{A}$. The simplest non-trivial class of examples that are relevant to this monograph are the *Lévy processes* where $(X(t), t \geq 0)$ is a process with stationary and independent increments taking values in $E = \mathbb{R}^d$. In this case (see e.g. Chapter 3 of [1] for details) the twice continuously differentiable functions of compact support $C_c^2(\mathbb{R}^d) \subseteq \text{Dom}(\mathcal{A})$ and for all $f \in C_c^2(\mathbb{R}^d), x \in \mathbb{R}^d$,

$$\begin{aligned}\mathcal{A}f(x) &= \sum_{i=1}^{d} b_i \partial_i f(x) + \sum_{i,j=1}^{d} a_{ij} \partial_i \partial_j f(x) \\ &+ \int_{\mathbb{R}^d - \{0\}} \left[f(x+y) - f(x) - \sum_{i=1}^{d} y_i \partial_i f(x) 1_{|y|<1} \right] \nu(dy).\end{aligned}$$

Here $b \in \mathbb{R}^d, a = (a_{ij})$ is a non-negative definite symmetric matrix and ν is a Lévy measure, i.e. $\int_{\mathbb{R}^d - \{0\}} \min\{1, |y|^2\} \nu(dy) < \infty$. The characteristics (b, a, ν) describe the drift, diffusion and jump intensity of the process (respectively.)

This result has a wide-reaching generalisation. In work of Phillipe Courrège in the mid-1960s it was shown that if $\mathcal{A}$ is the generator of any Feller process for which $C_c^2(\mathbb{R}^d) \subseteq \text{Dom}(\mathcal{A})$ then it essentially has the form given above, but the characteristics (b, a, ν) now exhibit functional dependence on $\mathbb{R}^d$ so in particular, ν is a *Lévy kernel* satisfying $\int_{\mathbb{R}^d - \{0\}} \min\{1, |y|^2\} \nu(x, dy) < \infty$ for each $x \in \mathbb{R}^d$. For a modern account of Courrège's work, the reader is directed to section 4.5 of [4]. To pick up one of the main themes in the monograph under review, we dualise and let ρ_t be the probability law at time t of the random variable $X(t)$, so that $\rho_t(B) = P(X(t) \in B)$ for each Borel set in $\mathbb{R}^d$. We then obtain the "Kolmogorov backwards equation"

$$\frac{d}{dt} \langle \rho_t, f \rangle = \langle \mathcal{A}^* \rho_t, f \rangle,$$

for each $f \in C_c^2(\mathbb{R}^d)$ where $\langle \rho_t, f \rangle := \int_{\mathbb{R}^d} f(x) \rho_t(dx)$ and $\mathcal{A}^*$ is the adjoint of $\mathcal{A}$ acting on the space $\mathcal{P}(\mathbb{R}^d)$ of probability measures on $\mathbb{R}^d$. A recent new development which has been initiated by the author of this monograph is to consider a non-linear generalisation of the Kolmogorov equation which has the form

$$\frac{d}{dt} \langle \rho_t, f \rangle = \Omega(\rho_t) f,$$

where Ω has the same structure as $\mathcal{A}$ but the characteristics (b, a, ν) are now functions defined on $\mathcal{P}(\mathbb{R}^d)$. It is the study of such equations which is the principal theme of this monograph. From now on I will refer to Feller processes that are obtained in this way as belonging to the *Kolokoltsov class.*

So far in this review I have focussed on the analytic approach to studying the Kolokoltsov class. But many Feller processes may be obtained as solutions of stochastic differential equations (SDEs) using the stochastic calculus that was introduced by K.Itô and his followers. A similar approach can be used to study processes in the Kolokoltsov class and here we need to utilise a non-linear stochastic integral of similar type to that studied by Carmona and Nualart in their monograph [3].

The book has a forty page introduction which begins with a discussion of the simplest possible example - a non-linear Markov chain and then gives brief accounts of some relevant concrete models including the Smoluchowski equation for coagulation (and its extension to include cluster formation), replicative dynamics for evolutionary games and the classical equations of kinetic theory (due to Vlasov, Boltzmann and Landau-Fokker-Planck, respectively.) The main part of the book is divided into three parts. Part 1 comprises four chapters on background to Markov process theory including the semigroup approach, the construction of non-linear stochastic integrals and associated stochastic differential equations (SDEs) and regularity results (including the case where the coefficients of the SDE are unbounded.) Part 2 is specifically dedicated to the study of the Kolokoltsov class. This is carried out both analytically (from the semigroup perspective) and probabilistically (using non-linear SDEs.) There are three chapters which deal with the issues of existence, uniqueness, well-posedness and smoothness with respect to initial data. The final chapter of this part describes applications to the Smoluchowski and Boltzmann equations.

Before describing the content of Part 3, we recall the three classes of Feller processes discussed above through the form of their generators. These are the Lévy processes whose characteristics are constant, those of Courrège type whose characteristics manifest spatial dependence, and finally those in the Kolokoltsov class where the characteristics are functions of measures. The first class is of great interest to probabilists (see e.g. the monograph [2]) but few Lévy processes directly describe any interesting dynamics for modelling purposes. On the other hand the second class has been extensively exploited in e.g. physics, engineering and economics where many phenomena are described as diffusions or jump-diffusions. Indeed mathematical finance would be lost without this technology. The third class is also likely to be of great value for modellers of complex interacting particle systems. One of the highlights of the book under review is the proof of a "law of large numbers". This is applicable to a number of concrete models such as the kinetic equations of statistical mechanics and models of coagulation. After suitable scaling it is shown that these models converge to a dynamics within the Kolokoltsov class, and the non-linear dynamics described above is now revealed to be a continuum limit as the number of particles goes to infinity. Furthermore the author also proves a "central limit theorem" which analyses fluctuations around the limit and shows that these are Gaussian (within a suitable infinite-dimensional framework.)

Part 3 is dedicated to applications. There are three chapters. The first two of these deal with the proofs of the law of large numbers and central limit theorem. A final chapter looks at some very interesting extensions such as a non-linear quantum dynamics suitable for studying quantum open systems and a curvilinear version of the well-known Ornstein-Uhlenback process.

This monograph is suitable for graduate students and researchers who have a good background in probability theory and analysis and have mastered such key techniques as martingales, stochastic calculus and weak convergence on the one hand and the analytic theory of semigroups on the other hand. Beyond that the book is quite self-contained and a number of useful technical notions such as variational derivatives of functions and measures and the Skorohod and Prokhorov topologies on the space of right continuous functions with left limits are dealt with in appendices which are highly illuminating in their own right.

This book is pioneering in developing a new and important type of dynamics for modelling complex stochastic systems. It deserves to be widely read.

References

[1] D.Applebaum, *Lévy Processes and Stochastic Calculus* (second edition), Cambridge University Press (2009).

[2] J. Bertoin, *Lévy Processes*, Cambridge University Press (1996).

[3] R. Carmona, D. Nualart, *Non-Linear Stochastic Integrators, Equations and Flows*, Gordon and Breach (1990).

[4] N. Jacob, *Pseudo-Differential Operators and Markov Processes: 1, Fourier Analysis and Semigroups*, World Scientific (2001).

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