Review of "Harmonic Analysis of Mean Periodic Functions on Symmetric Spaces and the Heisenberg Group" by Valery V. Volchkov and Vitaly V. Volchkov, Springer Monographs in Mathematics, Springer-Verlag London Ltd. (2009) ISBN 978-1-84882-532-1, pp. 671 £59-99 (Hardback)

A function f defined on the real line $\mathbb{R}$ is said to be *periodic* with period k if $f(x+k) = f(x)$ for all $x \in \mathbb{R}$. We can rewrite this is terms of convolution with a difference of Dirac δ-functions, which we understand in the sense of generalised functions or distributions. We then have that f is periodic if and only if

$$f * T = 0 \tag{0.1}$$

where $T(x) = \delta(x+k) - \delta(x)$. The main topic of this book is *mean periodic functions* and for a smooth f to be mean periodic we require that it satisfies the equation (0.1) for some distribution T having compact support. The study of this class of functions was initiated by Jean Delsarte in 1936 and developed further by Laurent Schwartz (the founder of the theory of distributions) in 1947. Just as periodic functions can be succinctly analysed using Fourier series, so mean periodic functions have more complicated infinite series expansions as "nonharmonic Fourier series."

The purpose of this treatise is to give a systematic account of mean periodic functions in more general spaces than $\mathbb{R}$ or $\mathbb{R}^n$ where the notion of distribution still has meaning. In fact we can define distributions on any smooth manifold but in order to have good global analogues of Fourier series we need to be more restrictive. One very rich line of development is to work on a symmetric space. A Riemannian manifold is a (globally) symmetric space if each point is fixed by an isometry that reverses geodesics passing through it. Examples include spheres, projective spaces and hyperbolic spaces. In fact every symmetric space is a homogeneous space G/K of a Lie group G quotiented by a closed subgroup K and this enables Lie group techniques to be exploited in the study of symmetric spaces. Fourier analysis on spheres in carried out by using spherical harmonics which are constructed using Legendre polynomials. More generally there is a beautiful theory of harmonic analysis on symmetric spaces which utilises generalisations of spherical harmonics called *spherical functions.* These were intensely studied by the great Indian mathematician Harish-Chandra in the 1950s. There are deep connections between spherical functions and the classical theory of special functions and many spherical functions can be expressed in terms of hypergeometric functions. These spherical functions are the key tools for developing the Fourier analysis of mean periodic functions in symmetric spaces

Another interesting context for harmonic analysis is the Heisenberg group. Here the underlying manifold is $\mathbb{R}^{n+1}$ which is equipped with a non-abelian group operation of similar type to that which arises in quantum mechanics and which underlies the well known Heisenberg uncertainty principle.

The book under review is a masterly treatise whose aim is to present the theory of mean periodic functions in symmetric spaces and on the Heisenberg group with particular emphasis on recent advances, many of which are due to the authors. The book is divided into four parts, each of which contains four to seven chapters and we don't meet mean periodic function until just over half way through. Part 1 is a very useful account of analysis in symmetric spaces while Part 2 gives an account of a key technique called transmutation whereby convolution equations on more complex spaces are "transmuted" into convolution equations on the real line which are, in principle, easier to analyse. The development of mean periodic functions can be found in Parts 3 and 4. Each part closes with a very interesting short section wherein the authors give the historical background for key results and also present some open problems.

This book is for experts in geometric analysis. It is a heroic achievement but requires considerable background on differential geometry, Lie groups, representation theory, distributions and special functions. Parts of it, especially the more general material in Part 1, might be of general interest to researchers in differential geometry, analysis and probability whose work wanders into symmetric spaces. It should certainly be in the library of every university where there is research in mathematics.

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