Second Quantised Representation of Mehler Semigroups Associated with Banach Space Valued Lévy processes

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Reproducing Kernel Hilbert Space (RKHS)

E is a real separable Banach space, E^* is its dual,

$\langle \cdot, \cdot \rangle$ is pairing $E \times E^* \to \mathbb{R}$.

$T \in \mathcal{L}(E^*, E)$ is

- *symmetric* if for all $a, b \in E^*$, $\langle Ta, b \rangle = \langle Tb, a \rangle$,
- *positive* if for all $a \in E^*$, $\langle Ta, a \rangle \geq 0$.

If T is positive and symmetric, $[\cdot, \cdot]$ is an inner product on $\mathrm{Im}(T)$, where

$$[Ta, Tb] = \langle Ta, b \rangle.$$

RKHS H_T is closure of $\mathrm{Im}(T)$ in associated norm.

Inclusion $\iota_T : \mathrm{Im}(T) \to E$ extends to a continuous injection $\iota_T : H_T \to E$.

$$T = \iota_T \circ \iota_T^*.$$

Infinite Divisibility in Banach Spaces

μ a Borel measure on E. Reversed measure $\widetilde{\mu}(E) = \mu(-E)$. μ *symmetric* if $\widetilde{\mu} = \mu$.

μ a (Borel) probabililty measure on E Its *Fourier transform/ characteristic function* is the mapping $\widehat{\mu} : E^* \to \mathbb{C}$ defined for $a \in E^*$ by:

$$\widehat{\mu}(a) = \int_E e^{i\langle x, a \rangle} \mu(dx).$$

A measure $\nu \in \mathcal{M}(E)$ is a *symmetric Lévy measure* if it is symmetric and satisfies

(i) $\nu(\{0\}) = 0$,
(ii) The mapping from E^* to $\mathbb{R}$ given by

$$a \to \exp\left\{\int_E [\cos(\langle x, a \rangle) - 1]\nu(dx)\right\}$$

is the characteristic function of a probability measure on E.

$\nu \in \mathcal{M}(E)$ is a *Lévy measure* if $\nu + \tilde{\nu}$ is a symmetric Lévy measure.

If ν is a Lévy measure on E, the mapping from E^* to $\mathbb{C}$ given by

$$a \to \exp\left\{\int_E [e^{i\langle x, a \rangle} - 1 - i\langle x, a \rangle 1_{B_1}(x)]\nu(dx)\right\}$$

is the characteristic function of a probability measure on E.

We say that a probability measure μ on E is *infinitely divisible* if for all $n \in \mathbb{N}$, μ has a convolution nth root μ_n.

Equivalently for all $n \in \mathbb{N}$ there exists a probability measure μ_n on E such that $\widehat{\mu}(a) = (\widehat{\mu_n}(a))^n$ for all $a \in E^*$.

Theorem (Lévy-Khintchine)

A probability measure $\mu \in \mathcal{M}_1(E)$ *is infinitely divisible if and only if there exists* $x_0 \in E$*, a positive symmetric operator* $R \in \mathcal{L}(E^*, E)$ *and a Lévy measure* ν *on* E *such that for all* $a \in E^*$*,*

$$\widehat{\mu}(a) = e^{\eta(a)},$$

where

$$\begin{aligned}\eta(a) &= i\langle x_0, a\rangle - \frac{1}{2}\langle Ra, a\rangle \\ &+ \int_E (e^{i\langle y,a\rangle} - 1 - i\langle y, a\rangle 1_{B_1}(y))\nu(dy).\end{aligned}$$

The triple (x_0, R, ν) is called the *characteristics* of the measure ν and η is known as the *characteristic exponent.*

Covariance Operators

A probability measure μ on E has *uniformly weak second order moments* if

$$\sup_{||a||\leq 1} \int_E |\langle x, a\rangle|^2 \mu(dx) < \infty.$$

In this case, there exists a *covariance operator* $Q \in \mathcal{L}(E^*, E)$ which is positive and symmetric:

$$\langle Qa, b\rangle = \int_E \langle x, a\rangle\langle x, b\rangle\mu(dx) - \left(\int_E \langle x, a\rangle\mu(dx)\right)\left(\int_E \langle x, b\rangle\mu(dx)\right).$$

Associated RKHS is H_Q.

If μ is infinitely divisible with characteristics (x_0, R, ν) and has uniformly weak second order moments:

$$Qa = Ra + \int_E \langle x, a\rangle x\, \nu(dx).$$

Mehler Semigroups

Let $(\mu_t, t \geq 0)$ be a family of probability measures on E with $\mu_0 = \delta_0$ and $(S(t), t \geq 0)$ be a C_0-semigroup on E. Define $T_t : B_b(E) \to B_b(E)$ by

$$T_t f(x) = \int_E f(S(t)x + y)\mu_t(dy).$$

$(T_t, t \geq 0)$ is a semigroup, i.e. $T_{t+s} = T_t T_s$ if and only if $(\mu_t, t \geq 0)$ is a *skew-convolution semigroup*, i.e.

$$\mu_{t+u} = \mu_u * S(u)\mu_t$$

(where $S(u)\mu_t := \mu_t \circ S(u)^{-1}$.)

Note that $T_t : C_b(E) \to C_b(E)$ but it is not (in general) strongly continuous.

From now on, we assume that the skew-convolution semigroup $(\mu_t, t \geq 0)$ is *F-differentiable*, i.e. $a \in E^*, t \to \widehat{\mu_t}(a)$ is differentiable.

$$\text{Define } \xi(a) := \frac{d}{dt}\widehat{\mu_t}(a)\bigg|_{t=0}.$$

Then

$$\widehat{\mu_t}(a) = e^{\eta_t(a)} := \exp\left\{\int_0^t \xi(S(u)^* a)du\right\}.$$

From this it follows that μ_t is infinitely divisible for all $t \geq 0$.

Furthermore ξ is the characteristic exponent of an infinitely divisible probability measure ρ with characteristics (b, R, ν) (say) and the characteristics of μ_t are (b_t, R_t, ν_t) where:

$$x_t = \int_0^t S(r)bdr + \int_0^t \int_E S(r)y(1_B(S(r)y) - 1_B(y))\nu(dy)dr,$$

$$R(t) = \int_0^t S(r)RS(r)^*)dr$$

$$\nu_t(A) = \int_0^t \nu(S(r)^{-1}A)dr.$$

see Bogachev, Röckner, Schmuland, PTRF **105**, 193 (1996); Furhman,Röckner, Pot. Anal. **12**, 1 (2000)

If ρ has covariance Q then μ_t has covariance

$$\begin{aligned} Q_t &= \int_0^t S(r)QS(r)^* dr \\ &= R_t + \int_0^t \int_E \langle S(r)y, a\rangle S(r)y\nu(dy) \end{aligned}$$

from which it follows that

$$Q_{t+s} = Q_t + S(t)Q_sS(t)^*.$$

Let H_t be RKHS of Q_t. Then

$$S(r)H_t \subseteq H_{t+r} \text{ and } ||S(r)||_{\mathcal{L}(H_t, H_{t+r})} \leq 1.$$

see J. van Neerven, JFA **155**, 495 (1998)

Contraction Properties

Theorem

If $(T_t, t \geq 0)$ is a Mehler semigroup then T_t is a contraction from $L^2(E, \mu_{t+u})$ to $L^2(E, \mu_u)$ for all $u \geq 0$.

Proof. For each $f \in L^2(E, \mu_u)$,

$$\begin{aligned} ||T_t f||^2_{L^2(\mu_u)} &= \int_E |T_t f(x)|^2 \mu_u(dx) \\ &= \int_E \left| \int_E f(S(t)x + y)\mu_t(dy) \right|^2 \mu_u(dx) \\ &\leq \int_E \int_E |f(S(t)x + y)|^2 \mu_t(dy)\mu_u(dx) \\ &= \int_E |f(x)|^2 (\mu_t * S(t)\mu_u)(dx) \\ &= \int_E |f(x)|^2 \mu_{u+t}(dx) = ||f||^2_{L^2(\mu_{t+u})} \end{aligned}$$

□

Lévy Driven OU Processes

Let A be the infinitesmal generator of the semigroup $(S(t), t \geq 0)$.

Let $(X(t), t \geq 0)$ be an E-valued Lévy process.

Consider the linear SPDE with additive noise:

$$dY(t) = AY(t) + dX(t) \;\; ; \;\; Y(0) = Y_0$$

Unique solution is *generalised Ornstein-Uhlenbeck process*:

$$Y(t) = S(t)Y_0 + \int_0^t S(t-u)dX(u).$$

Transition semigroup $T_t f(x) = \mathbb{E}(f(Y(t))|Y_0 = x)$ is a Mehler semigroup.

Skew convolution semigroup μ_t is law of $\int_0^t S(t-u)dX(u) \stackrel{d}{=} \int_0^t S(u)dX(u)$

$\mathbb{E}(e^{i\langle X(t), a\rangle}) = e^{t\xi(a)}$ for all $a \in E^*, t \geq 0$.

Second Quantisation

"First quantisation is a mystery, second quantisation is a functor."
Ed. Nelson

H a complex Hilbert space. $\Gamma(H)$ is symmetric Fock space over H.

$$\Gamma(H) := \bigoplus_{n=0}^{\infty} H_s^{(n)}$$

$H^{(0)} = \mathbb{C}, H^{(1)} = H, H^{(n)}$ is n fold symmetric tensor product

Exponential vectors $\{e(f), f \in H\}$ are linearly independent and total where

$$e(f) = \left(1, f, \frac{f \otimes f}{\sqrt{2!}}, \ldots, \frac{f^{\otimes^n}}{\sqrt{n!}}, \ldots\right), \quad \langle e(f), e(g)\rangle = e^{\langle f,g\rangle}.$$

$$n\text{-particle vector } f^{\otimes^n} = \frac{1}{\sqrt{n!}} \frac{d}{da} e(af)\Big|_{a=0}.$$

Second quantisation of contractions: If C is a contraction in H then $\Gamma(C)$ is a contraction in $\Gamma(H)$ where

$$\Gamma(C)e(f) = e(Cf).$$

Gaussian Spaces μ a Gaussian measure on E (i.e. infinitely divisible with characteristics $(0, R, 0)$).

Isometric embedding $\Phi : H_R \to L^2(E, \mu)$ given by continuous extension of

$$\Phi(Ra) = \langle \cdot, a\rangle, \quad \text{for } a \in E^*$$

For $h \in H_R$, define $K_h \in L^2(E, \mu)$ by

$$K_h(x) = \exp\left\{\Phi_h(x) - \frac{1}{2}||h||^2\right\}.$$

Canonical isomorphism between $\Gamma(H_R)$ and $L^2(E, \mu)$ given by $e(h) \to K_h$.

Gaussian Mehler Semigroups

Assume each μ_t Gaussian, covariance R_t, RKHS H_t.

Since $S(t)$ is a contraction from H_u to $H_{u+t} \Rightarrow S(t)^*$ is a contraction from H_{u+t} to H_u.

Recall that T_t is a contraction from $L^2(E, \mu_{t+u})$ to $L^2(E, \mu_u)$.

$$T(t) = \Gamma(S(t)^*).$$

J van Neerven, JFA **155**, 495 (1998)

A. Chojnowska-Michalik and B. Goldys, J.Math. Kyoto Univ. **36**, 481 (1996)

Non-Gaussian Mehler Semigroups

Assume μ infinitely divisible with $\widehat{\mu}(a) = e^{\eta(a)}$ for $a \in E^*$. We need analogues of exponential vectors. For each $a \in E^*$ define $K_a \in L^2_{\mathbb{C}}(E, \mu)$ by

$$K_a(x) = e^{i\langle x,a\rangle - \eta(a)}.$$

Theorem

The set $\{K_a, a \in E^*\}$ is total in $L^2_{\mathbb{C}}(E, \mu)$.

Proof. Let $\psi \in L^2_{\mathbb{C}}(E, \mu)$ be such that for all $a \in E^*$, $\int_E K_a(x)\overline{\psi(x)}\mu(dx) = 0$. Then $\int_E e^{i\langle x,a\rangle}\mu_\psi(dx) = 0$, where $\mu_\psi(dx) := \overline{\psi(x)}\mu(dx)$ is a complex measure. It follows by injectivity of the Fourier transform that $\mu_\psi = 0$ and hence $\psi = 0$ (a.e.) as was required. □

Theorem

The set $\{K_a, a \in E^*\}$ is linearly independent in $L^2_{\mathbb{C}}(E, \mu)$.

Proof. Let $a_1, \ldots a_n \in E^*$ be distinct and $c_1, \ldots, c_n \in \mathbb{C}$ for some $n \in \mathbb{N}$ and assume that $\sum_{i=1}^n c_i K_{a_i} = 0$.

Define $\tilde{c}_i := e^{-\eta(a_i)} c_i$ for $1 \leq i \leq n$ and replace x by tx where $t \in \mathbb{R}$.

Then we have $\sum_{i=1}^n \tilde{c}_i e^{it\langle x, a\rangle} = 0$ for all $t \in \mathbb{R}$. Let $t = 0$ to see that $\sum_{i=1}^n \tilde{c}_i = 0$.

Now differentiate r times with respect to t (where $1 \leq r \leq n - 1$) and then put $t = 0$. This yields $\sum_{i=1}^n \tilde{c}_i \langle x, a_i\rangle^r = 0$.

We have a system of n linear equations in $\tilde{c}_1, \ldots \tilde{c}_n$ and it has a non-zero solution if and only if

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \langle x, a_1\rangle & \langle x, a_2\rangle & \cdots & \langle x, a_n\rangle \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ \langle x, a_1\rangle^{n-1} & \langle x, a_2\rangle^{n-1} & \cdots & \langle x, a_n\rangle^{n-1} \end{vmatrix} = 0.$$

This ia a Vandermonde determinant and so the equation simplifies to

$$\prod_{1 \leq i,j \leq n} (\langle x, a_i\rangle - \langle x, a_j\rangle) = 0.$$

Hence there exists k, l with $1 \leq k, l \leq n$ such that $\langle x, a_k - a_l\rangle = 0$ for all $x \in E$. It follows that $a_k = a_l$ and this is a contradiction. So we must have $\tilde{c}_1 = \tilde{c}_2 = \cdots = \tilde{c}_n = 0$. Since $e^{-\eta(a)} \neq 0$ for all $a \in E^*$, we deduce that $c_1 = c_2 = \cdots = c_n = 0$, as was required. □

non-Gaussian Second Quantisation

Let $T \in \mathcal{L}(E^*)$. We define its *second quantisation* $\Gamma(T)$ to be the densely defined linear operator with domain $\mathcal{E} = \text{lin span}\{K_a, a \in E^*\}$ defined by linear extension of the prescription

$$\Gamma(T)K_a = K_{Ta}.$$

The following properties are straightforward to verify:

- $\Gamma(T)$ is closeable with $\mathcal{E} \subseteq \Gamma(T)^*$ and $\Gamma(T)^* = \Gamma(T^*)$,
- If $T_1, T_2 \in \mathcal{L}(E^*)$ then $\Gamma(T_1 T_2) = \Gamma(T_1)\Gamma(T_2)$.

In the case where $\mu = \mu_t$, we write $K_{t,a}$ instead of K_a.

The Main Result

Theorem

Let $(\mu_t, t \geq 0)$ be an F-differentiable skew convolution semigroup. For all $t, u > 0$

$$T_t = \Gamma(S(t)^*_{t+u \to u}).$$

Proof. For all $a \in E^*, x \in E$

$$\begin{aligned} T_t K_{t+u,a}(x) &= \int_E K_{t+u,a}(S(t)x + y)\mu_t(dy) \\ &= e^{-\eta_{t+u}(a)} e^{i\langle S(t)x, a\rangle} \int_E e^{i\langle y, a\rangle} \mu_t(dy) \\ &= e^{-\eta_{t+u}(a)} e^{\eta_t(a)} e^{i\langle S(t)x, a\rangle}. \end{aligned}$$

However

$$\begin{aligned}\eta_t(a) - \eta_{t+u}(a) &= -\int_t^{t+u} \xi(S(r)^* a) dr \\ &= -\int_0^u \xi(S(r)^* S(t)^* a) dr \\ &= -\eta_u(S(t)^* a).\end{aligned}$$

From this we see that

$$\begin{aligned}T_t K_{t+u,a}(x) &= e^{i\langle x, S(t)^* a\rangle - \eta_u(S(t)^* a)} \\ &= K_{u,S(t)^* a}(x),\end{aligned}$$

and the required result follows. □

Some comments:
We did not assume that μ_t has second moments and made no use of a RKHS. So our second quantisation

$$\Gamma : \mathcal{L}(E^*) \to \text{closeable lin.ops on } L^2_{\mathbb{C}}(E, \mu) \text{ preserving } \mathcal{E}.$$

If we assume that μ_t has second moments for all t then $S(t)^*$ is a contraction from H_{t+u} to H_u.

$T_t = \Gamma(S(t)^*)$ is a contraction from $L^2(E, \mu_{t+u})$ to $L^2(E, \mu_u)$.

Invariant Measures

λ is an invariant measure for the Mehler semigroup $(T_t, t \geq 0)$ if and only if for all $t \geq 0$

$$\lambda = \mu_t * S(t)\lambda.$$

If λ exists it is infinitely divisible (operator self-decomposable.)

e.g. if $\mu_\infty = \text{weak-}\lim_{n\to\infty} \mu_t$ exists it is an invariant measure.

If E is a Hilbert space and we are in the Ornstein-Uhlenbeck case:

A.Chojnowska-Michalik Stochastics, **21** 251 (1987)

e.g. Assume $(S_t, t \geq 0)$ is *exponentially stable* i.e. $||S(t)|| \leq Me^{-\lambda t}$ for $M \geq 1, \lambda > 0$. Then necessary and sufficient conditions for unique invariant measure are

$$\int_0^\infty \int_E (||S(t)y||^2 \wedge 1)\nu(dy)dt < \infty,$$

$$\lim_{t\to\infty} \int_0^t \int_E S(r)y(1_B(S(r)y) - 1_B(y))\nu(dy)dr \text{ exists.}$$

Further if $\int_0^\infty \int_E ||S(t)y||^2 \nu(dy)dt < \infty$, then μ_∞ has covariance operator

$$\begin{aligned}Q_\infty &= \int_0^\infty S(r)QS(r)^* dr \\ &= R_\infty + \int_0^\infty \int_E \langle S(r)y, a\rangle S(r)y\nu(dy)\end{aligned}$$

We get RKHS H_∞ with for all $t \geq 0$

$$S(t)H_\infty \subseteq H_\infty \text{ and } ||S(t)||_{\mathcal{L}(H_\infty)} \leq 1.$$

Also T_t is a contraction in $L^2_{\mathbb{C}}(E, \mu_\infty)$ and $T_t = \Gamma(S(t)^*)$.

The Chaos Approach in the non-Gaussian case.

Based on work by

S.Peszat, JFA **260**, 3457 (2011)

$(\Omega, \mathcal{F}, P)$ is a probability space. Let Π be a Poisson random measure defined on a measurable space $(E, \mathcal{B})$ with intensity measure λ. Let $\mathbb{Z}_+(E)$ be the non-negative integer valued measures on $(E, \mathcal{B})$. Regard Π as a random variable on Ω taking values in $\mathbb{Z}_+(E)$ by

$$\Pi(\omega)(E) = \Pi(E, \omega)$$

Let P_π be the law of Π and for $F \in L^2(P_\pi), \xi \in \mathbb{Z}_+(E)$ define the "Malliavin derivative":

$$D_y F(\xi) = F(\xi + \delta_y) - F(\xi)$$

Define $T^n : L^2(P_\pi) \to L^2_{\text{symm}}(E^n, \lambda^n)$ by

$$(T^n F)(y_1, \ldots, y_n) = \mathbb{E}(D^n_{y_1,\ldots,y_n} F(\Pi)).$$

Chaos expansion

$$\mathbb{E}(F(\Pi)G(\Pi)) = \mathbb{E}(F(\Pi))\mathbb{E}(G(\Pi)) + \sum_{n=1}^{\infty} \frac{1}{n!} \langle T^n F, T^n G \rangle_{L^2(E^n, \lambda^n)}$$

from which it follows that

$$F(\Pi) = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(T^n F),$$

where I_n is usual multiple Itô integral w.r.t. compensator $\tilde{\Pi} := \Pi - \lambda$. So here $L^2(P_\pi) = \Gamma(L^2(E, \lambda))$.

see G.Last, M.Penrose, PTRF **150**, 663 (2011)

Peszat: If E is a Hilbert space, $R \in \mathcal{L}(E)$, define $\rho_R^{(n)} \in \mathcal{L}(L^2(E^n, \lambda^n))$ by

$$\rho_R^{(n)} f(y_1, \ldots, y_n) = f(Ry_1, \ldots, Ry_n).$$

Second quantisation: $\Gamma_0(R) : L^2(P_\pi) \to L^2(P_\pi)$,

$$\Gamma_0(R)F = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(\rho_R^{(n)}(T^n F)).$$

$$T_t = \Gamma_0(S(t)^*)$$

Connecting The Two Approaches in the non-Gaussian Case

For all $t \geq 0$ let $S_t := [0, t) \times E$.
Let Π be a Poisson random measure defined on $[0, \infty) \times E$ so that Π_t has intensity measure λ_t.
The natural filtration of $\Pi_t(\cdot) := \Pi(t, \cdot)$ is denoted $(\mathcal{F}_t, t \geq 0)$.
For $t \geq 0, f \in L^2(S_t, \lambda_t)$ define the process $(X_f(t), t \geq 0)$ by

$$X_f(t) = \int_0^t \int_E f(s, x) \tilde{\Pi}(ds, dx).$$

$$\mathbb{E}(|X_f(t)|^2) = ||f||^2_{L^2(S_t, \lambda_t)} < \infty.$$

$$\mathbb{E}(e^{iX_f(t)}) = e^{\eta_f(t)},$$

where

$$\eta_f(t) = \int_{S_t} (e^{if(s,x)} - 1 - if(s, x)) \lambda(ds, dx).$$

Define the process $(M_f(t), t \geq 0)$ by

$$M_f(t) = \exp\{iX_f(t) - \eta_f(t)\}.$$

Then $(M_f(t), t \geq 0)$ is a square-integrable martingale with

$$dM_f(t) = \int_{S_t} (e^{if(s,x)} - 1) M_f(s-) \tilde{\Pi}(ds, dx),$$

and for all $t \geq 0$,

$$\mathbb{E}(|M_f(t)|^2) = \exp\left\{\int_{S_t} |e^{if(s,x)} - 1|^2 \lambda(ds, dx)\right\} \qquad (1.1)$$

Lemma

For all $t \geq 0$,

$$\mathbb{E}(|M_f(t)|^2) \leq e^{||f||^2_{L^2(S_t, \lambda_t)}}.$$

Proof. Using the well known inequality $1 - \cos(y) \leq \frac{y^2}{2}$ for $y \in \mathbb{R}$

$$\begin{aligned}\mathbb{E}(|M_f(t)|^2) &= \exp\left\{2\int_{S_t} (1 - \cos(f(s,x)))\lambda(ds, dx)\right\} \\ &\leq \exp\left\{\int_0^t \int_H f(s,x)^2 \lambda(ds, dx)\right\} \\ &= e^{||f||^2_{L^2(S_t, \lambda_t)}}.\end{aligned}$$

Now let $(Y_f(t), t \geq 0)$ be the Doléans-Dade exponential which is the unique solution of the stochastic differential equation

$$dY_f(t) = Y_f(t-) dX_f(t),$$

with initial condition $Y_f(0) = 1$ (a.s.)

$$Y_f(t) = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(f^{\otimes^n}) \text{ and } \mathbb{E}(|Y_f(t)|)^2 = e^{||f||^2_{L^2(S_t, \lambda_t)}}.$$

Let $\mathcal{K}(t)$ be the linear span of $\{M_f(t), f \in L^2(S_t, \lambda_t)$.

Let $\mathcal{L}(t)$ be the linear span of $\{Y_f(t), f \in L^2(S_t, \lambda_t)$.

Both sets are total in $L^2(\Omega, \mathcal{F}_t, P)$.
The map $C : \mathcal{K}(t) \to \mathcal{L}(t)$ which takes each $M_f(t)$ to $Y_f(t)$ extends to an invertible linear operator on $L^2(\Omega, \mathcal{F}_t, P)$ which we continue to denote by C.
Note that C is a contraction by above lemma.

Now assume that μ_t has uniformly finite weak second order moments and is and for each $a \in E^*, t \geq 0$ define

$$f_a \in L^2(S_t, \lambda) \text{ by } f_a(s,x) = \langle x, a\rangle 1_{[0,t)}(s) \text{ for each } 0 \leq s \leq t, x \in E.$$

Then we have $M_f(t) = M_{t,a}$ where

$$M_{t,a}(x) = \exp\left\{i\int_E \langle x, a\rangle \tilde{\Pi}(t, dx) - \eta_t(x)\right\},$$

$$\eta_t(x) = \int_E (e^{i\langle x,a\rangle} - 1 - i\langle x, a\rangle)\lambda_t(dx).$$

Then $M_{t,a}$ is precisely the image of $K_{t,a}$ in $L^2(\Omega, \mathcal{F}_t, P)$ under the natural embedding of $L^2(E, \mu_t)$ into that space. From now on we will identify these vectors.

For each $t \geq 0$, we write the Doléans-Dade exponential $Y_a(t)$ when $f = f_a$ as above.

Theorem

For each $S \in \mathcal{L}(E^*)$

$$\Gamma(S) = C^{-1}\Gamma_0(S^*)C,$$

Proof. For each $a \in E^*, t \geq 0$,

$$\begin{aligned} \Gamma(S)C^{-1}Y_a(t) &= \Gamma(S)K_{t,a} \\ &= K_{t,Sa} \\ &= C^{-1}Y_{Sa}(t) \\ &= C^{-1}\Gamma_0(S^*)Y_a(t), \end{aligned}$$

and the result follows. □