Asymptotic Stability of Stochastic Differential Equations Driven by Lévy Noise

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Abstract

Using key tools such as Itô's formula for general semimartingales, Kunita's moment estimates for Lévy-type stochastic integrals, and the exponential martingale inequality, we find conditions under which the solutions to the stochastic differential equations (SDEs) driven by Lévy noise are stable in probability, almost surely and moment exponentially stable.

Keywords; stochastic differential equation, Lévy noise, Poisson random measure, Brownian motion, almost sure asymptotic stability, moment exponential stability, Lyapunov exponent.

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1 Introduction

There has recently been increasing interest in stochastic differential equations (SDEs) driven by noise that has discontinuous jumps. The case where the noise is obtained from a Lévy process via its Lévy-Itô decomposition into a Brownian motion (continuous part) and independent Poisson random measure (jump part) has attracted particular interest and Applebaum [1] is a recent monograph devoted to this topic. Indeed such SDEs are finding a considerable range of applications including financial economics (see e.g. Cont and Tankov [4] and references therein), stochastic filtering and control (Øksendal and Sulem [17]), CARMA time series models (Brockwell [2]) and stochastic resonance in non-linear signal processing (Patel and Kosko [18]).

The long-time asymptotic behaviour of solutions to SDEs is very important. In particular we would like to know if a stationary solution exists and to be able to estimate the rate of convergence to it. In the literature particular attention has focussed on the case where there is a trivial solution and Lyapunov exponents can be calculated. In the case of SDEs driven by Brownian motion, the linear case was first investigated by

Khasminski [8]. The extension to non-linear SDEs driven by continuous semimartingales and also to stochastic delay and more general stochastic functional differential equations has been extensively studied by X. Mao in a series of books and articles (see [12, 14, 15] and references therein).

The theory is much less well-developed in the case where the driving noise has jumps. Mao and Rodkina [16] have studied a class of SDEs driven by semimartingales with jumps but the conditions they impose are not easily applied in the Lévy noise case. An extensive study of linear SDEs driven by Lévy noise has been carried out by Li, Dong and Situ [11] while Grigoriu [5] has studied some special cases (both linear and non-linear) for SDEs driven by compound Poisson processes (see also Grigoriu and Samorodnitsky [5]).

The purpose of this paper is to extend Mao's techniques to the case of non-linear SDEs driven by Lévy noise, i.e. a Brownian motion and an independent (and separately coupled) Poisson random measure. We focus on the results given in Chapter 4 of [15] and extend these to the Lévy case. We will omit proofs when these are straightforward generalisations of the Brownian motion case and concentrate on those results where more careful analysis is needed. We mainly study two types of stochastic stability in this paper - almost sure exponential stability and moment exponential stability. Full definitions of these and related concepts are given in section 2. In section 3 we present our results on almost sure exponential stability while moment exponential stability is tackled in section 4.

In general there is no obvious relation between exponential and almost sure stability (see Kozin [9] p.107). However it is possible when moment stability holds to deduce almost sure stability under some additional conditions as shown for the Brownian motion case by Mao [15]. In the last section we extend this result for SDEs driven by Lévy noise.

Finally we remark that all our results extend easily to suitable SDEs with time-dependent coefficients as in Mao [15].

Notation. Throughout this paper $\mathbb{R}^+ := [0, \infty)$. The open ball of radius $c > 0$ that is centred on the origin is denoted by B_c and $\widehat{B}_c := B_c - \{0\}$. $\mathcal{M}_{d,m}(\mathbb{R})$ is the space of all real-valued $d \times m$ matrices and if $A \in \mathcal{M}_{d,m}(\mathbb{R})$ then $||A|| := \left(\sum_{i=1}^{d}\sum_{j=1}^{m}|A_{ij}A_{ji}|\right)^{\frac{1}{2}}$. The Euclidean norm of a vector x is denoted by $|x|$ throughout.

2 Preliminaries

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), P)$ be a filtered probability space that satisfies the usual hypotheses of completeness and right continuity. Assume that we are given an m-dimensional standard $\mathcal{F}_t$-adapted Brownian motion $B = (B(t), t \geq 0)$ with each $B(t) = (B^1(t), \ldots, B^m(t))$ and an independent $\mathcal{F}_t$-adapted Poisson random measure N defined on $\mathbb{R}^+ \times (\mathbb{R}^d - \{0\})$ with compensator $\tilde{N}$ and intensity measure ν, where we assume that ν is a Lévy measure so that $\tilde{N}(dt, dy) := N(dt, dy) - \nu(dy)dt$ and $\int_{\mathbb{R}^d-\{0\}}(|y|^2 \wedge 1)\nu(dy) < \infty$. We call the pair (B, N) a *Lévy noise.*

Let $0 \leq t_0 \leq T \leq \infty$. Assume that the mappings $f : \mathbb{R}^d \to \mathbb{R}^d$, $g : \mathbb{R}^d \to \mathcal{M}_{d,m}(\mathbb{R})$,

$H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ satisfy the usual global Lipschitz and growth conditions (see Applebaum [1] Theorem 6.2.3 p. 304). We consider SDEs driven by Lévy noise of the form

$$dx(t) = f(x(t-))dt + g(x(t-))dB(t) + \int_{|y|<c} H(x(t-), y)\tilde{N}(dt, dy) \qquad \text{on } t \geq t_0 \quad (1)$$

with initial value $x(t_0) = x_0$, such that $x_0 \in \mathbb{R}^d$. Here $c \in (0, \infty]$ is the maximum allowable jump size. We remark that all the results in the sequel can alternatively be established under local Lipschitz conditions and a suitable monotone growth condition as in Siakalli [19] p.52 (see also Mao [15], section 2.3).

We assume that $f(0) = 0$, $g(0) = 0$, $H(0, y) = 0$ for all $|y| < c$ then (1) has a unique solution $x(t) = 0$ for all $t \geq t_0$ corresponding to the initial value $x(t_0) = 0$, which is called the *trivial solution.*

We will consider three types of stability, these being stability in probability, almost sure and moment exponential stability.

Definition 2.1 The trivial solution of (1) is said to be *stable in probability* if for every pair of $\varepsilon \in (0, 1)$ and $r > 0$, there exists a $\delta = \delta(\varepsilon, r, t_0)$ such that

$$P\{|x(t)| < r \text{ for all } t \geq t_0\} \geq 1 - \varepsilon \quad (2)$$

whenever $|x_0| < \delta$.

Definition 2.2 The trivial solution of (1) is said to be *almost surely exponentially stable* if

$$\limsup_{t\to\infty} \frac{1}{t} \log |x(t)| < 0 \qquad a.s. \quad (3)$$

for all $x_0 \in \mathbb{R}^d$. The quantity in the left hand side of (3) is called the *sample Lyapunov exponent.*

Definition 2.3 Assume that $p > 0$. The trivial solution of (1) is said to be *pth moment exponentially stable* if there is a pair of constants $\lambda > 0$ and $C > 0$ such that

$$E[|x(t)|^p] \leq C|x_0|^p \exp(-\lambda(t - t_0)) \qquad \text{for all } t \geq t_0 \quad (4)$$

for all $x_0 \in \mathbb{R}^d$. In this case we call the quantity $\limsup_{t\to\infty} \frac{1}{t} \log(E(|x(t)|^p))$ the *pth moment Lyapunov exponent.*

In this paper we will need Kunita's estimates (see Kunita [10]) for the solution of an SDE of the form (1).

Theorem 2.4 *(Kunita) For all $p \geq 2$, there exists $C(p, t) > 0$ such that for each*

$t > t_0 \geq 0$,

$$E\left[\sup_{t_0\leq s\leq t}|x(s)|^p\right] \leq C(p,t)\Bigg\{|x_0|^p + E\left[\int_{t_0}^t |f(x(r-))|^p dr\right] + E\left[\int_{t_0}^t \|g(x(r-))\|^p dr\right] + E\left[\int_{t_0}^t \left(\int_{|y|<c} |H(x(r-),y)|^2\nu(dy)\right)^{\frac{p}{2}} dr\right] + E\left[\int_{t_0}^t \int_{|y|<c} |H(x(r-),y)|^p \nu(dy)dr\right]\Bigg\} \quad (5)$$

where $x(t_0) = x_0 \in \mathbb{R}^d$ is the initial condition.

The proof can be found in Kunita [10] pp.332-335 (see also Corollary 4.2.44 in Applebaum [1], second edition).

We will also need the following technical exponential martingale inequality for stochastic integrals involving both Brownian motion and Poisson random measures. In the former case the integrand lives in the space $\mathcal{P}_2(T)$ which is the linear space of all predictable mappings $F : [0,T]\times\Omega \to \mathbb{R}^d$ for which $P\left[\int_0^T |F(t)|^2 dt < \infty\right] = 1$ and in the latter case we require integrands that belong to the space $\mathcal{P}_2(T,E)$ which comprises predictable mappings (in the sense of Applebaum [1] Chapter 4) $H : [0,T]\times E\times\Omega \to \mathbb{R}^d$ which satisfy $P\left[\int_0^T \int_E |H(s,y)|^2\nu(dy)ds < \infty\right] = 1$ where E is a given Borel set in $\mathbb{R}^d - \{0\}$.

Theorem 2.5 *(Exponential Martingale Inequality)*
Let T, α, β be any positive numbers. Assume that $g \in \mathcal{P}_2(T)$ and $H \in \mathcal{P}_2(T,E)$. Then

$$P\Big[\sup_{0\leq t\leq T}\Big\{\int_0^t g(s)\,dB(s) - \frac{\alpha}{2}\int_0^t \big|g(s)\big|^2 ds + \int_0^t\int_{|y|<c} H(s,y)\,\tilde{N}(ds,dy) - \frac{1}{\alpha}\int_0^t\int_{|y|<c}\Big[\exp\big(\alpha H(s,y)\big) - 1 - \alpha H(s,y)\Big]\nu(dy)ds\Big\} > \beta\Big] \leq \exp\big(-\alpha\beta\big). \quad (6)$$

For the proof see Applebaum [1] second edition pp 287-288 or Siakalli [19].

In this paper we will mainly be concerned with almost sure asymptotic stability and moment exponential stability. However we will include one result on stability in probability. For this we need the linear operator $\mathcal{L} : C^2(\mathbb{R}^d) \to C(\mathbb{R}^d)$ associated to the SDE (1).

$$(\mathcal{L}V)(x) = f^i(x)(\partial_i V)(x) + \frac{1}{2}[g(x)g(x)^T]^{ik}(\partial_i\partial_k V)(x) + \int_{|y|<c}\left[V(x+H(x,y)) - V(x) - H^i(x,y)(\partial_i V)(x)\right]\nu(dy) \quad (7)$$

where $V \in C^2(\mathbb{R}^d)$, $x \in \mathbb{R}^d$.

Theorem 2.6 *Let $c \in (0,\infty)$ and let B_h be the open ball of radius $h \geq 2c$ that is centred on the origin in $\mathbb{R}^d$. Assume that there exists a positive definite function*

$V \in C^2(B_h;\mathbb{R}^+)$ such that

$$\mathcal{L}V(x) \leq 0$$

for all $x \in B_h$. Then the trivial solution of (1) is stable in probability.

We omit the proof as it is very similar to the Brownian motion case presented in Mao [15], Chapter 4, Theorem 2.2. For full details see Siakalli [19], section 3.3. We will however point out that there is a slight variation in the statement of Theorem 2.6 from the Brownian motion case which involves the jump size c. This is because a stopping time argument in Mao [15] needs to be slightly adapted to take account of the jumps of the solution. We also point out that positive definiteness here is in the sense of Lyapunov, i.e. we require that $V(0) = 0$ and that $V(x) \geq \kappa(|x|)$ for all $x \in B_h$ for some continuous non-decreasing function $\kappa : \mathbb{R}^+ \to \mathbb{R}^+$.

3 Almost surely asymptotic stability

In order to be able to develop the theory in this section we need the following technical inequality.

Lemma 3.1 If $x, y \in \mathbb{R}^d$, $x, x+y \neq 0$ then

$$\frac{1}{|x+y|} - \frac{1}{|x|} + \frac{\langle x, y\rangle}{|x|^3} \leq \frac{2|y|}{|x|^2}\left(\frac{|y|+|x|}{|x+y|}\right).$$

Proof: Using the Cauchy-Schwarz inequality we find that

$$\begin{aligned}
\frac{1}{|x+y|} - \frac{1}{|x|} + \frac{\langle x, y\rangle}{|x|^3} &= \frac{|x|^3 - |x|^2.|x+y| + |x+y|.\langle x, y\rangle}{|x|^3.|x+y|} \\
&\leq \frac{|x|^3 - |x|^2.|x+y| + (|x+y|).|x|.|y|}{|x|^3.|x+y|} \\
&\leq \frac{|x|^2 - |x|.(|x|-|y|) + |y|.(|x|+|y|)}{|x|^2.|x+y|} \\
&= \frac{|y|^2 + 2|x|.|y|}{|x|^2.|x+y|} \leq \frac{2|y|}{|x|^2}.\left(\frac{|y|+|x|}{|x+y|}\right).
\end{aligned}$$

□

The main result of this section depends critically on the result of the lemma below which is a generalization of Mao's work in the Brownian motion case (see [13, 15] pp. 280-281 and pp. 120-121 respectively). We will prove that under some conditions the solution of (1) can never reach the origin provided that $x_0 \neq 0$.

Assumption 3.2 We suppose that H is always such that

$$\nu\left\{y \in \widehat{B}_c, \text{there exists } x \neq 0 \text{ such that } x + H(x,y) = 0\right\} = 0.$$

We require that Assumption 3.2 holds for the rest of this section.

Lemma 3.3 Assume that for any $\theta > 0$ there exists $K_\theta > 0$, such that

$$|f(x)| + \|g(x)\| + 2\int_{|y|<c} |H(x,y)| \left(\frac{|x| + |H(x,y)|}{|x + H(x,y)|} \right) \nu(dy) \le K_\theta |x| \ \text{ if } |x| \le \theta. \tag{8}$$

If $x_0 \neq 0$ then

$$P\left(x(t) \neq 0 \ \text{ for all } t \ge t_0\right) = 1. \tag{9}$$

Proof: Assume that (9) is false. This implies that for some $x_0 \neq 0$ there will be a stopping time τ with $P(\tau < \infty) > 0$ when the solution will be zero for the first time:

$$\tau = \inf\{t \ge t_0 : |x(t)| = 0\}.$$

Since the paths of x are almost surely right continuous with left limits (see e.g. Applebaum [1] Theorem 6.2.3, p.304) there exists $T > t_0$ and $\theta > 1$ such that $P(B) > 0$ where

$$B = \left\{\omega \in \Omega : \tau(\omega) \le T \text{ and } |x(t)(\omega)| \le \theta - 1 \ \text{ for all } t_0 \le t \le \tau(\omega)\right\}.$$

Let $V(x) = |x|^{-1}$. If $0 < |x| \le \theta$ it follows from (7) and Lemma 3.1 that

$$\mathcal{L}V(x) \quad \le \quad \frac{|f(x)|}{|x|^2} + \frac{\|g(x)\|^2}{|x|^3} + 2\int_{|y|<c} \left[\frac{|H(x,y)|}{|x|^2} \left(\frac{|H(x,y)| + |x|}{|x + H(x,y)|} \right) \right] \nu(dy) \tag{10}$$

Applying (8) to (10) then

$$\mathcal{L}V(x) \le \alpha V(x) \qquad \text{if} \quad 0 < |x| \le \theta$$

where α is a positive constant.

Now define the following family of stopping times

$$\tau_\varepsilon = \inf\{t \ge t_0 : |x(t)| \le \varepsilon \text{ or } \ |x(t)| \ge \theta\}$$

for each $0 < \varepsilon < |x_0|$. Following exactly the same arguments as in Mao [13, 15] pp. 280-281 and pp. 120-121 respectively we have that

$$E\left[e^{-\alpha(\tau_\varepsilon \wedge T - t_0)} V(x(\tau_\varepsilon \wedge T))\right] \quad \le \quad V(x_0).$$

If $\omega \in B$, then $\tau_\varepsilon(\omega) \le T$ and $|x(\tau_\varepsilon(\omega))| \le \varepsilon$. Then,

$$\begin{aligned} E\left[e^{-\alpha(T-t_0)}\varepsilon^{-1} 1_B\right] \quad &\le \quad E\left[e^{-\alpha(\tau_\varepsilon - t_0)} |x(\tau_\varepsilon(\omega))|^{-1} 1_B\right] = E\left[e^{-\alpha(\tau_\varepsilon \wedge T - t_0)} V(x(\tau_\varepsilon \wedge T)) 1_B\right] \\ &\le \quad E\left[e^{-\alpha(\tau_\varepsilon \wedge T - t_0)} V(x(\tau_\varepsilon \wedge T))\right] \le V(x_0). \end{aligned}$$

Hence,

$$P(B) \le \varepsilon\, e^{\alpha(T-t_0)} |x_0|^{-1}, \quad \text{for all } \varepsilon \ge 0.$$

Now let $\varepsilon \to 0$. Then it follows that $P(B) = 0$ which contradicts the definition of the set B and the required result follows. $\square$

Remark 3.4 Condition (8) in Lemma 3.3 seems quite complicated. We will now show that there is a natural class of mappings H for which this is satisfied, at least in the case $d = 1$. To begin suppose that we can find a mapping H_1 for which

$$\int_{|y|<c} |H_1(x,y)|\nu(dy) < K_\theta |x|, \text{ for all } x \in \mathbb{R}.$$

Now let $A = \{(x,y) \in \mathbb{R}^2 : x \geq 0, H_1(x,y) \geq 0\} \cup \{(x,y) \in \mathbb{R}^2 : x \leq 0, H_1(x,y) \leq 0\}$ and so $A^c = \{(x,y) \in \mathbb{R}^2 : x \geq 0, H_1(x,y) < 0\} \cup \{(x,y) \in \mathbb{R}^2 : x \leq 0, H_1(x,y) > 0\}$.

Define $H(x,y) = \big(1_A(x,y) - 1_{A^c}(x,y)\big)H_1(x,y)$.
Hence,

$$|H_1(x,y)| = |H(x,y)| \text{ and } |x + H(x,y)| = |x| + |H_1(x,y)|.$$

Then we find that

$$\int_{|y|<c} |H(x,y)|.\left(\frac{|x| + |H(x,y)|}{|x + H(x,y)|}\right)\nu(dy) = \int_{|y|<c} |H_1(x,y)|\nu(dy) < K_\theta|x|, \text{ for all } x \in \mathbb{R}.$$

To construct specific examples of mappings of the form H_1 we can take e.g. $H_1(x,y) = H_2(x)y^2$ where $\frac{H_2(x)}{x}$ is bounded.

For the next two results, we require that the following local boundedness constraint on the jumps holds:

Assumption 3.5 For all bounded sets M in $\mathbb{R}^d$,

$$\sup_{x\in M}\sup_{0<|y|<c} |H(x,y)| < \infty.$$

In the sequel conditions for almost sure exponential stability of the trivial solution of (1) will be obtained. First we need a useful technical result.

Let $V \in C^2(\mathbb{R}^d;\mathbb{R}^+)$ be such that $V(x) \neq 0$ for every $x \in \mathbb{R}^d$. Define the following processes $I_1 = (I_1(t), t \geq t_0)$, $I_2 = (I_2(t), t \geq t_0)$ and $I = (I(t), t \geq t_0)$ where for each $t \geq t_0$

$$I_1(t) = \int_{t_0}^t \int_{|y|<c} \left(\frac{V\big(x(s-) + H(x(s-),y) - V(x(s-))\big)}{V(x(s-))} - \frac{H^i(x(s-),y)}{V(x(s-))}\partial_i V(x(s-))\right)\nu(dy)ds, \tag{11}$$

$$I_2(t) = \int_{t_0}^t \int_{|y|<c} \left(\log\left(\frac{V\big(x(s-) + H(x(s-),y)\big)}{V(x(s-))}\right) + 1 - \frac{V\big(x(s-) + H(x(s-),y)\big)}{V(x(s-))}\right)\nu(dy)ds, \tag{12}$$

$$I(t) = \int_{t_0}^{t} \int_{|y|<c} \left(\log \left(\frac{V\big(x(s-) + H(x(s-),y)\big)}{V(x(s-))} \right) - \frac{H^i(x(s-),y)}{V(x(s-))} \partial_i V(x(s-)) \right) \nu(dy)ds. \tag{13}$$

Note that for each $t \geq t_0$, $I(t) = I_1(t) + I_2(t)$.

Lemma 3.6 Let $I_1 = (I_1(t), t \geq t_0)$, $I_2 = (I_2(t), t \geq t_0)$ and $I = (I(t), t \geq t_0)$ be defined for each $t \geq t_0$ as in (11), (12), (13) respectively. Then for each $t \geq t_0$, it holds that

$$(i)\ |I_1(t)| < \infty, \quad (ii)\ |I(t)| < \infty, \quad \text{and} \quad (iii)\ |I_2(t)| < \infty \qquad a.s.$$

Proof: (i) Following Kunita's arguments in [10] pp. 317, by using a Taylor's series expansion with integral remainder term (see Burkill [3], Theorem 7.7) we obtain for each $y \in \hat{B}_c$ and $x \in \mathbb{R}^d$

$$V\big(x + H(x,y)\big) - V(x) - H^i(x,y)\partial_i V(x) = \int_0^1 \partial_i \partial_j V\big(x + \theta H(x,y)\big)(1-\theta)d\theta H^i(x,y)H^j(x,y).$$

Hence,

$$\begin{aligned} &|I_1(t)| \\ &\leq \int_{t_0}^{t} \int_{|y|<c} \left| \frac{V\big(x(s-) + H(x(s-),y)\big) - V(x(s-)) - H^i(x(s-),y)\partial_i V(x(s-))}{V(x(s-))} \right| \nu(dy)ds \\ &\leq \frac{1}{2} \int_{t_0}^{t} \int_{|y|<c} \left| \sup_{0 \leq \theta \leq 1} \frac{\partial_i \partial_j V\big(x(s-) + \theta H(x(s-),y)\big)}{V(x(s-))} \right| \big|H^i(x(s-),y)H^j(x(s-),y)\big| \nu(dy)ds. \end{aligned} \tag{14}$$

For each $z \in \mathbb{R}^d$, $y \in \hat{B}_c$, $1 \leq i,j \leq d$, define

$$f_{ij}^V(z,y) = \sup_{0 \leq \theta \leq 1} \frac{\partial_i \partial_j V\big(z + \theta H(z,y)\big)}{V(z)}.$$

By Assumption 3.5 it follows that

$$\sup_{t_0 \leq s \leq t} \sup_{0<|y|<c} |f_{ij}^V(x(s-),y)| < \infty \qquad a.s.$$

Using the Cauchy-Schwarz inequality it follows from (14) that

$$\begin{aligned}|I_1(t)| &\leq \frac{1}{2}\sup_{t_0\leq s\leq t}\sup_{0<|y|<c}|f_{ij}^V(x(s-),y)|\int_{t_0}^t\int_{|y|<c}\left|H^i(x(s-),y)H^j(x(s-),y)\right|\nu(dy)ds\\ &\leq \frac{1}{2}\left(\sum_{i,j=1}^d\sup_{t_0\leq s\leq t}\sup_{0<|y|<c}|f_{ij}^V(x(s-),y)|^2\right)^{\frac{1}{2}}\int_{t_0}^t\int_{|y|<c}|H(x(s-),y)|^2\nu(dy)ds<\infty,\end{aligned} \tag{15}$$

almost surely. (ii) follows by the same arguments as in (i) and (iii) is then immediate. □

The following is a generalization of Mao's work [15] Chapter 4, Theorem 3.3 pp. 121.

Theorem 3.7 *Let $V\in C^2(\mathbb{R}^d;\mathbb{R}^+)$ and let $p>0, c_1>0, c_2\in\mathbb{R}, c_3\geq 0$ and $c_4\geq 0$ be such that for all $x\neq 0$*

$$\begin{array}{ll}(i) & c_1|x|^p\leq V(x),\\ (ii) & \mathcal{L}V(x)\leq c_2V(x),\\ (iii) & \left|(\partial V(x))^T g(x)\right|^2\geq c_3\big(V(x)\big)^2,\\ (iv) & \displaystyle\int_{|y|<c}\left[\log\left(\frac{V(x+H(x,y))}{V(x)}\right)-\frac{V\big(x+H(x,y)\big)-V(x)}{V(x)}\right]\nu(dy)\leq -c_4.\end{array}$$

Then

$$\limsup_{t\to\infty}\frac{1}{t}\log|x(t)|\leq -\frac{c_3+2c_4-2c_2}{2p}\qquad a.s. \tag{16}$$

and furthermore if $c_3>2c_2-2c_4$, then the trivial solution of (1) is almost surely exponentially stable for all $x_0\in\mathbb{R}^d$.

Remark 3.8 Using the logarithmic inequality $\log(x)\leq x-1$ for $x>0$ then

$$\int_{|y|<c}\left[\log\left(\frac{V(x+H(x,y))}{V(x)}\right)-\frac{V\big(x+H(x,y)\big)-V(x)}{V(x)}\right]\nu(dy)\leq 0.$$

Hence condition (iv) in Theorem 3.7 is a reasonable constraint to require.

Proof: For $x_0=0$, then $x=0$ hence (16) holds trivially. For the rest of the proof we assume that $x_0\neq 0$. We first assume that (8) holds. Due to Lemma 3.3, then $x(t)\neq 0$ for all $t\geq t_0$ almost surely. Apply Itô's formula to $Z(t)=\log(V(x(t)))$. Then for each

$t \geq t_0$,

$$\begin{aligned}
\log(V(x(t))) &= \log(V(x_0)) + \int_{t_0}^{t} \frac{1}{V(x(s-))} \partial_i V(x(s-)) \left[f^i(x(s-))ds + g^{ij}(x(s-))dB_j(s) \right] \\
&+ \frac{1}{2} \int_{t_0}^{t} \Big[\frac{1}{V(x(s-))} \partial_i \partial_k V(x(s-))[g(x(s-))g(x(s-))^T]^{ik} \\
&\qquad - \frac{1}{(V(x(s-)))^2} \left| (\partial V(x(s-)))^T g(x(s-)) \right|^2 \Big] ds \\
&+ \int_{t_0}^{t} \int_{|y|<c} \Big[\log\Big(V\big(x(s-) + H(x(s-), y)\big)\Big) - \log\big(V(x(s-))\big) \Big] \tilde{N}(ds, dy) \\
&+ \int_{t_0}^{t} \int_{|y|<c} \Big[\log\Big(V\big(x(s-) + H(x(s-), y)\big)\Big) - \log\big(V(x(s-))\big) \\
&\qquad - \frac{1}{V(x(s-))} \partial_i V(x(s-)) H^i(x(s-), y) \Big] \nu(dy) ds.
\end{aligned} \tag{17}$$

Note that the last integral in (17) is almost surely finite by Lemma 3.6.

Now using the linear operator $\mathcal{L}$ defined in (7) we obtain

$$\begin{aligned}
\log(V(x(t))) &\leq \log(V(x_0)) + \int_{t_0}^{t} \frac{\mathcal{L}V(x(s-))}{V(x(s-))} ds + M(t) \\
&- \frac{1}{2} \int_{t_0}^{t} \frac{1}{(V(x(s-)))^2} \left| (\partial V(x(s-)))^T g(x(s-)) \right|^2 ds + I_2(t),
\end{aligned} \tag{18}$$

where for each $t \geq t_0$

$$\begin{aligned}
M(t) &= \int_{t_0}^{t} \frac{1}{V(x(s-))} \partial_i V(x(s-)) g^{ij}(x(s-)) dB_j(s) \\
&\qquad + \int_{t_0}^{t} \int_{|y|<c} \log\Big(\frac{V\big(x(s-) + H(x(s-), y)\big)}{V(x(s-))} \Big) \tilde{N}(ds, dy).
\end{aligned}$$

We now apply the exponential martingale inequality (6) for $T = n$, $\alpha = \varepsilon$ and $\beta = \varepsilon n$ where $\varepsilon \in (0,1)$ and $n \in \mathbb{N}$. Then for every integer $n \geq t_0$, we find that

$$\begin{aligned}
P\Bigg[\sup_{t_0 \leq t \leq n} \Bigg\{ & M(t) - \frac{\varepsilon}{2} \int_{t_0}^{t} \frac{1}{(V(x(s-)))^2} \left| (\partial V(x(s-)))^T g(x(s-)) \right|^2 ds \\
& - \frac{1}{\varepsilon} \int_0^t \int_{|x|<c} \Bigg[e^{\log\left(\frac{V\left(x(s-)+H(x(s-),y)\right)}{V(x(s-))} \right)^{\varepsilon}} - 1 \\
& \qquad - \varepsilon \log\left(\frac{V\big(x(s-) + H(x(s-), y)\big)}{V(x(s-))} \right) \Bigg] \nu(dy) ds \Bigg\} > \varepsilon n \Bigg] \leq e^{-\varepsilon^2 n}.
\end{aligned}$$

Since $\sum_{n=1}^{\infty} e^{-\varepsilon^2 n} < \infty$ an application of the Borel-Cantelli lemma and elementary probability calculations yields that

$$P\Bigg[\liminf_{n\to\infty}\Bigg\{\sup_{t_0\le t\le n}\Bigg(M(t)-\frac{\varepsilon}{2}\int_{t_0}^{t}\frac{1}{(V(x(s-)))^2}\left|(\partial V(x(s-)))^T g(x(s-))\right|^2 ds$$
$$-\frac{1}{\varepsilon}\int_0^t\int_{|x|<c}\left(\frac{V\big(x(s-)+H(x(s-),y)\big)}{V(x(s-))}\right)^{\varepsilon}-1$$
$$-\varepsilon\log\left(\frac{V\big(x(s-)+H(x(s-),y)\big)}{V(x(s-))}\right)\nu(dy)ds\Bigg)\le\varepsilon n\Bigg\}\Bigg]=1.$$

Hence for almost all $\omega\in\Omega$ there is a random integer $n_0=n_0(\omega)$ such that for $n\ge n_0$, $t_0\le t\le n$,

$$\begin{aligned}M(t) \le\ & \frac{\varepsilon}{2}\int_{t_0}^{t}\frac{1}{(V(x(s-)))^2}\left|(\partial V(x(s-)))^T g(x(s-))\right|^2 ds+\varepsilon n\\ &+\frac{1}{\varepsilon}\int_{t_0}^{t}\int_{|y|<c}\Bigg[\left(\frac{V(x(s-)+H(x(s-),y))}{V(x(s-))}\right)^{\varepsilon}-1\\ &\qquad-\varepsilon\log\left(\frac{V(x(s-)+H(x(s-),y))}{V(x(s-))}\right)\Bigg]\nu(dy)ds. \qquad (19)\end{aligned}$$

Substituting (19) into (18) and using conditions (ii) and (iii) it follows immediately that

$$\begin{aligned}&\log(V(x(t)))\\ &\le\log(V(x_0))-\frac{1}{2}[(1-\varepsilon)c_3-2c_2](t-t_0)+\varepsilon n\\ &+\int_{t_0}^{t}\int_{|y|<c}\left[\log\left(\frac{V\big(x(s-)+H(x(s-),y)\big)}{V(x(s-))}\right)+1-\frac{V\big(x(s-)+H(x(s-),y)\big))}{V(x(s-))}\right]\nu(dy)ds\\ &+\frac{1}{\varepsilon}\int_{t_0}^{t}\int_{|y|<c}\Bigg[\left(\frac{V(x(s-)+H(x(s-),y))}{V(x(s-))}\right)^{\varepsilon}-1\\ &\qquad-\varepsilon\log\left(\frac{V(x(s-)+H(x(s-),y))}{V(x(s-))}\right)\Bigg]\nu(dy)ds \qquad (20)\end{aligned}$$

for $n\ge n_0$, $t_0\le t\le n$.

Fix $x\in\mathbb{R}^d$ and define for $y\in\hat{B}_c$, $h_\varepsilon(y)=\frac{1}{\varepsilon}\left|\left(\frac{V(x+H(x,y))}{V(x)}\right)^{\varepsilon}-1-\varepsilon\log\left(\frac{V(x+H(x,y))}{V(x)}\right)\right|$. We easily deduce that $\left(\frac{V(x+H(x,y))}{V(x)}\right)^{\varepsilon}-1-\varepsilon\log\left(\frac{V(x+H(x,y))}{V(x)}\right)\ge 0$ for all $y\in\hat{B}_c$, by using the elementary inequality $e^b-1-b\ge 0$ for $b\in\mathbb{R}$. Since $\varepsilon\in(0,1)$ then we can use the inequality $b^c<1+c(b-1)$ for $0<c<1$ and $b>0$ (see Hardy, Littlewood and

Pólya [7] pp. 40) to deduce that for all $y \in \hat{B}_c$,

$$\begin{aligned} h_\varepsilon(y) &\leq \frac{1}{\varepsilon}\left[1 + \varepsilon\left(\frac{V(x+H(x,y))}{V(x)} - 1\right) - 1 - \varepsilon\log\left(\frac{V(x+H(x,y))}{V(x)}\right)\right] \\ &\leq \frac{V(x+H(x,y))}{V(x)} - 1 - \log\left(\frac{V(x+H(x,y))}{V(x)}\right). \end{aligned} \tag{21}$$

Now let $\varepsilon \to 0$. Using (21) and Lemma 3.6 (iii) we apply the dominated convergence theorem to deduce that for all $t \geq t_0$

$$\begin{aligned} &\lim_{\varepsilon\to 0}\int_{t_0}^t \int_{|y|<c} \frac{1}{\varepsilon}\Bigg[\left(\frac{V(x(s-)+H(x(s-),y))}{V(x(s-))}\right)^\varepsilon - 1 \\ &\qquad\qquad - \varepsilon\log\left(\frac{V(x(s-)+H(x(s-),y))}{V(x(s-))}\right)\Bigg]\nu(dy)ds \\ &= \int_{t_0}^t \int_{|y|<c} \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\left(\frac{V(x(s-)+H(x(s-),y))}{V(x(s-))}\right)^\varepsilon - 1\right] \\ &\qquad\qquad - \log\left(\frac{V(x(s-)+H(x(s-),y))}{V(x(s-))}\right)\nu(dy)ds \\ &= 0. \end{aligned} \tag{22}$$

Hence by (22) for $n \geq n_0$, $t_0 \leq t \leq n$, (20) becomes

$$\begin{aligned} \log(V(x(t))) &\leq \log(V(x_0)) - \frac{1}{2}(c_3 - 2c_2)(t - t_0) \\ &\quad + \int_{t_0}^t \int_{|y|<c} \Bigg[\log\left(\frac{V\big(x(s-)+H(x(s-),y)\big)}{V(x(s-))}\right) + 1 \\ &\quad - \frac{V\big(x(s-)+H(x(s-),y)\big)}{V(x(s-))}\Bigg]\nu(dy)ds. \end{aligned} \tag{23}$$

Now substituting condition (iv) into (23), we see that for almost all $\omega \in \Omega$, $t_0 + n - 1 \leq t \leq t_0 + n$, $n \geq n_0$

$$\frac{1}{t}\log(V(x(t))) \leq -\frac{t-t_0}{2t}(c_3 - 2c_2) + \frac{\log(V(x(t_0)))}{t_0 + n - 1} - \frac{t - t_0}{t}c_4.$$

Now applying condition (i), the required result follows. In the case where (8) fails to hold we may assume without loss of generality that $H \neq 0$ and that the process $x(t)$ hits the origin infinitely many times (with probability one). Define an increasing sequence of stopping times $(T_n, n \in \mathbb{N})$ by $T_1 = \inf\{t > t_0, x(t) = 0\}$ and for $n > 1, T_n = \inf\{t > T_{n-1}, x(t) = 0\}$. We now argue as above but with $x(t)$ replaced throughout by $y(t)$ where

$$y(t) = x(t)1_{[t_0,T_1)}(t) + \sum_{n=1}^{\infty} x(t)1_{(T_n,T_{n+1})}(t). \qquad \square$$

4 Moment Exponential Stability

The main aim of this section is to introduce criteria for the solution of an SDE driven by Lévy noise to be moment exponentially stable and to derive a relation between moment and almost sure exponential stability.

Theorem 4.1 *Let $p, \alpha_1, \alpha_2, \alpha_3$ be positive constants. If $V \in C^2(\mathbb{R}^d; \mathbb{R}^+)$ satisfies*

(i) $\alpha_1 |x|^p \le V(x) \le \alpha_2 |x|^p$,

(ii) $\mathcal{L}V(x) \le -\alpha_3 V(x)$,

for all $x \in \mathbb{R}^d$, then

$$E\left[|x(t)|^p\right] \le \frac{\alpha_2}{\alpha_1} |x_0|^p \exp(-\alpha_3(t - t_0)) \qquad \text{for all } t \ge t_0 \tag{24}$$

for all $x_0 \in \mathbb{R}^d$. As a result the trivial solution of (1) is pth moment exponentially stable under conditions (i) and (ii) and the pth moment Lyapunov exponent should not be greater than $-\alpha_3$.

The proof is omitted as it is a straightforward extension of the Brownian motion case as can be found in [15] Chapter 4, Theorem 4.4 pp. 130. We will however give a simple (linear) example to confirm that the conditions (i) and (ii) can be verified in the jump case. We take $d = 1$ and also $c = 1$. Let $V(x) = x^2$ so that (i) is automatically satisfied with $p = 2$. Now choose $f(x) = bx$ where $b \in \mathbb{R}$, $g(x) = x$ and $H(x, y) = xy$. Then (7) yields $\mathcal{L}V(x) = \left(2b + 1 + \int_{|y|<1} |y|^2 \nu(dy)\right) V(x)$ and so (ii) is satisfied provided b is chosen to satisfy $b \le -\frac{1}{2}\left(1 + \int_{|y|<1} |y|^2 \nu(dy)\right)$.

We note that if the hypotheses of Theorem 4.1 hold then the trivial solution of (1) is almost surely exponential stable as can be seen by taking $c_3 = c_4 = 0$ in Theorem 3.7. In the last part of the paper we will give conditions under which pth moment exponential stability for $p \ge 2$ always implies almost surely exponential stability for our equation.

Assumption 4.2 For all $2 \le q \le p$ and $K > 0$

$$\int_{|y|<c} \left|H(x, y)\right|^q \nu(dy) \le K \left|x\right|^q.$$

We require that Assumption 4.2 holds for the remainder of this section.

The following is an extension of Mao's work [15] Theorem 4.2, Chapter 4 pp. 128 that refers to SDEs driven by Brownian motion. We will generalize this result and give the relationship between the pth moment exponential stability and almost sure exponential stability for the trivial solution of (1).

Remark 4.3 Recall that in the context of stability theory we are always assuming that $f(0) = 0$ and $g(0) = 0$, hence from the Lipschitz conditions on f and g we deduce that for all $x \in \mathbb{R}^d$ there exists $L > 0$ such that $|f(x)| \le \sqrt{L}|x|$ and $\|g(x)\|^2 \le L|x|^2$.

Hence,

$$x^T f(x) \vee \|g(x)\|^2 \leq |x^T f(x)| \vee \|g(x)\|^2 \leq |x|.|f(x)| \vee \|g(x)\|^2 \leq L'|x|^2 \tag{25}$$

where $L' = \max\{\sqrt{L}, L\}$ and this will be used in the proof of the theorem below.

Theorem 4.4 *Assume that Assumption 4.2 holds. For $p \geq 2$, pth moment exponential stability, of the trivial solution to (1), implies almost sure exponential stability.*

Proof: Fix any $x_0 \neq 0$ on $\mathbb{R}^d$ and let $n \in \mathbb{N}$. Apply Ito's formula to $Z(t) = |x(t)|^p$, then using (25) and taking expectations it follows that

$$\begin{aligned}
&E\left[\sup_{t_0+n-1\leq t\leq t_0+n} |x(t)|^p\right] \\
&\leq E\left[|x(t_0+n-1)|^p\right] + \alpha \int_{t_0+n-1}^{t_0+n} E\left[|x(s-)|^p\right] ds \\
&\quad + E\left[\sup_{t_0+n-1\leq t\leq t_0+n} \int_{t_0+n-1}^{t} p|x(s-)|^{p-2} x(s-)^T g(x(s-)) dB(s)\right] + I_1 \quad (26)
\end{aligned}$$

where $\alpha = pL' + \frac{pL'}{2}[1 + (p-2)]$ and

$$\begin{aligned}
I_1 &= E\Bigg[\sup_{t_0+n-1\leq t\leq t_0+n} \Bigg\{ \int_{t_0+n-1}^{t} \int_{|y|<c} \Big(|x(s-) + H(x(s-), y)|^p - |x(s-)|^p\Big) \tilde{N}(ds, dy) \\
&\quad + \int_{t_0+n-1}^{t} \int_{|y|<c} \Big(|x(s-) + H(x(s-), y)|^p - |x(s-)|^p \\
&\qquad - p|x(s-)|^{p-2} x(s-)^T H(x(s-), y)\Big) \nu(dy) ds \Bigg\}\Bigg].
\end{aligned}$$

For the Brownian motion integral, as in Mao [15] pp. 129 we apply the Burkholder-Davis-Gundy inequality:

$$\begin{aligned}
&E\left[\sup_{t_0+n-1\leq t\leq t_0+n} \int_{t_0+n-1}^{t} p|x(s-)|^{p-2} x(s-)^T g(x(s-)) dB(s)\right] \\
&\leq \frac{1}{2} E\left[\sup_{t_0+n-1\leq t\leq t_0+n} |x(t-)|^p\right] + 16p^2 L' \int_{t_0+n-1}^{t_0+n} E[|x(s-)|^p] ds. \quad (27)
\end{aligned}$$

Applying Kunita's estimates for $f = 0$ and $g = 0$ it follows that

$$\begin{aligned}
I_1 &\leq \beta(p,t) \Bigg\{ E\left[\int_{t_0+n-1}^{t_0+n} \left(\int_{|y|<c} |H(x(s-), y)|^2 \nu(dy)\right)^{\frac{p}{2}} ds\right] \\
&\quad + E\left[\left(\int_{t_0+n-1}^{t_0+n} \int_{|y|<c} |H(x(s-), y)|^p \nu(dy) ds\right)\right] \Bigg\} \quad (28)
\end{aligned}$$

where $\beta(p,t)$ is a positive constant that depends only on t and p. Using Assumption 4.2 within (28), we obtain

$$I_1 \leq \gamma(p,t) E\left[\int_{t_0+n-1}^{t_0+n} |x(s-)|^p ds\right] \tag{29}$$

where $\gamma(p,t) = \beta(p,t)(K^{\frac{p}{2}} + K)$. Then (26) becomes

$$\begin{aligned} E\left[\sup_{t_0+n-1\leq t\leq t_0+n} |x(t)|^p\right] &\leq E[|x(t_0+n-1)|^p] + \frac{1}{2}E\left[\sup_{t_0+n-1\leq t\leq t_0+n} |x(t-)|^p\right] \\ &+ (c_1 + 16p^2L' + \gamma(p,t))\left(\int_{t_0+n-1}^{t_0+n} E[|x(s)|^p]ds\right). \end{aligned} \tag{30}$$

Rearranging, for $p \geq 2$

$$E\left[\sup_{t_0+n-1\leq t\leq t_0+n} |x(t)|^p\right] \leq 2E[|x(t_0+n-1)|^p] + \delta(p,t)\int_{t_0+n-1}^{t_0+n} E[|x(s)|^p]ds \tag{31}$$

where $\delta(p,t)$ is a positive constant depending on p and t. Now we argue as in Mao [15] pp. 129-130 and the required result follows. □

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