MAS331 METRIC SPACES

Background Information

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Office hours: Mon 11-12, Fri 11-12. Feel free to come at other times if this doesn't work for you, but please e-mail first to check that I'm there!

This module explores the concept of "distance" in an abstract mathematical framework. As well as capturing familiar notions of distance between two points in one, two or higher dimensions we will also be able to make sense of the distance between two continuous functions and this is very important in both pure and applied mathematics. This course falls into the area of mathematics that we call *analysis* and it follows on naturally from MAS221 "Analysis". Some of the key ideas that were introduced in that module - such as *convergence* and *continuity* will be extended to the more general setting of metric spaces. Another theme that will be explored in the course is the more practically oriented one of convergence of iteration schemes. Towards the end of the course we'll prove the "contraction mapping theorem" which gives criteria that guarantee convergence. This has applications to many areas of mathematics, including existence and uniqueness of solutions to differential equations.

Metric spaces give evidence of the power of abstraction in mathematics. The study of metric spaces can also be seen as a stepping stone towards two of the great subjects in contemporary mathematics – *functional analysis* which is where metric spaces meet vector spaces in an infinite dimensional setting, and *topology* - the most general context where continuity makes sense.

Assessment. The course will be assessed by one $2\frac{1}{2}$ hour exam at the end of semester one. You should answer 4 questions out of 5.

Weekly practice questions will be set on the material and solutions posted on the web. You are strongly encouraged to attempt these. You learn mathematics by doing it, not by listening or reading.

There will be three (non-assessed) assignments for you to hand in. These will be marked and returned to you for feedback.

There will be no lectures in Week 7, Reading Week.

Outline Syllabus

1. Metric spaces (4 lectures)

Distance functions. Definition of metric space. Review of suprema and infima. Examples including $\mathbb{R}^n$ and function spaces. Some basic properties of metrics. Subspaces. Closed balls and open balls.

2. Convergence of sequences (3 lectures)

Definition in terms of N and ϵ. Examples. Basic properties: uniqueness of limit, equivalence of $x_n \to x$ with $d(x_n, x) \to 0$ in $\mathbb{R}$. Convergence in $\mathbb{R}^2$ means convergence componentwise. Convergence in function spaces and pointwise convergence.

3. Closed and open sets (2 lectures)

Definition of closed sets and open sets. Behaviour under intersections and unions. Examples.

4. Continuity (2 lectures)

Continuity in terms of sequences and in terms of ϵ and δ. Relation with closed sets and open sets. Examples.

5. Cauchy sequences and completeness (2 lectures)

'Internal' tests for convergence, without knowledge of the limit. Cauchy sequences. Completeness. Bolzano–Weierstrass. Examples including $\mathbb{R}^n$ and function spaces.

6. Iteration and Contraction (4 lectures)

Iteration as a method to solve problems. Examples in $\mathbb{R}$, $\mathbb{R}^2$. Fixed points of iterations. Discussion of what should be required to guarantee convergence of iterative procedures to fixed points. Contractions. Examples. The Contraction Mapping Principle. An application to linear algebra. A differential criterion for a function to be a contraction. Functions with the property that repeated application gives a contraction. Application to existence of solution of differential equations. Examples.

7. Compactness (3 lectures)

Definition using subsequences. Compactness and continuity. Examples. Compact subsets of Euclidean spaces. Equivalent formulations in terms of total boundedness and completeness, and in terms of the Heine–Borel property.

Books

The book by Victor Bryant is the closest in spirit to this course. There are multiple copies in the Information Commons.

Victor Bryant, *Metric spaces, iteration and application*, Cambridge University Press (1985)

W.A. Sutherland, *Introduction to Metric and Topological Spaces*, Clarendon Press, Oxford (1975)

N.B.Haaser, J.A. Sullivan, *Real Analysis*, Dover (1991) (*Chapter 4 is most relevant to this course.*)

N.L.Carothers, *Real Analysis*, Cambridge University Press (1999)

E.Kreysig, *Introductory Functional Analysis with Applications*, John Wiley and Sons Ltd (1988) (*Chapter 1 is most relevant to this course.*)