Covariant Mehler Semigroups in Hilbert Space

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Dedicated to the memory of J.T.Lewis

Abstract

We find necessary and sufficient conditions for a generalised Mehler semigroup to be covariant under the action of a locally compact group. These are then applied to implement "noise reduction" for Hilbert-space valued Ornstein-Uhlenbeck processes driven by Lévy processes.

1 Introduction

Generalised Mehler semigroups are beautiful objects which have attracted the attention of both analysts and probabilists. They are semigroups of linear operators $(T(t), t \geq 0)$ acting on the space of bounded continuous functions on a real separable Hilbert space which are built from two components:

- a C_0-semigroup $(S(t), t \geq 0)$ acting in H with generator J,
- a family $(\mu_t, t \geq 0)$ of probability measures on H satisfying the skew-convolution property $\mu_{t+r} = \mu_t * (\mu_r \circ (S(t)^{-1})$,

through the formula

$$(T(t)f)(x) = \int_H f(S(t)x + y)\mu_t(dy).$$

From a probabilistic point of view, they arise as the transition semigroups of H-valued Ornstein-Uhlenbeck processes $(Y(t), t \geq 0)$ which are driven by

H-valued Lévy processes $(Z(t), t \geq 0)$ (i.e. processes with stationary and independent increments) through the stochastic differential equation:

$$dY(t) = JY(t)dt + dZ(t).$$

When H is infinite-dimensional, such equations have been extensively studied in the Gaussian case where Z is a standard Brownian motion (see e.g. [6]), and have been extended to the more general Lévy case in [5] and [2]. In the "classical case" where J is a negative scalar and H is finite dimensional, the use of Lévy noise has attracted recent attention through applications to volatility models in option pricing [3], [16] and branching processes with immigration [15].

From the analytic viewpoint, extensive work on Mehler semigroups has been carried out by M.Röckner and his collaborators ([4], [9], [12], [13], [18]). In particular, they have shown that when invariant measures exist, the generators can be constructed as pseudo-differential operators acting in a suitable L^p space [12], while in [18], the strong Feller property is proved and the Harnack, Poincaré and logarithmic Sobolev inequalities established. Applications of generalized Mehler semigroups to measure-valued catalytic branching processes have been developed in [7].

From a modelling viewpoint, a Mehler semigroup (or Ornstein-Uhlenbeck process) describes the interaction of a system, which may itself be highly complex, with a noisy environment. It is natural to seek to simplify the problem by exploiting symmetry, when this is present. To this end, in this paper we assume that the Hilbert space H carries a unitary representation of a group G and we ask for conditions under which the Mehler semigroup commutes with this group action. Our main result is that necessary and sufficient conditions for this are that the semigroup $(S(t), t \geq 0)$ itself commutes with the group action and the probabilities $(\mu_t, t \geq 0)$ are left invariant. When G acts irreducibly, this forces $S(t)$ to be trivial, so we have a "classical" Ornstein-Uhlenbeck process and the driving noise is simplified. In the non-compact case, we have extensive "noise reduction" in that the only admissible (i.e. suitably invariant) driving Lévy processes are those of pure jump type.

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Notation $\mathbb{R}^+ = [0, \infty)$. If T is a topological space, then $\mathcal{B}(T)$ denotes its Borel σ-algebra. If H is a real separable Hilbert space, $B_b(H)$ is the space of bounded Borel measurable real-valued functions on H and $\mathcal{L}(H)$ is the $*$-algebra of all bounded linear operators on H. If M is a non-empty subset of $\mathcal{L}(H)$, then M'' is the smallest von Neumann algebra containing M. I is the identity operator in $\mathcal{L}(H)$. The domain of a linear operator T acting in

H is denoted as Dom(T). A mapping from $\mathbb{R}^+$ to H is càdlàg if it is right continuous and has a left limit at every point.

If $x \in H, \delta_x$ denotes Dirac mass at x which is the probability measure on $\mathcal{B}(H)$ defined by $\delta_x(A) = 1$ if $x \in A$, and $\delta_x(A) = 0$ if $x \notin A$. $\mu_1 * \mu_2$ denotes the convolution of Borel measures μ_1 and μ_2 defined on H.

If $(\Omega, \mathcal{F}, P)$ is a probability space and $X : \Omega \to H$ is a random variable, the law of X is the probability measure p_X defined on $\mathcal{B}(H)$ by $p_X(A) = P(X \in A)$. If X and Y are two such random variables, we write $X \stackrel{d}{=} Y$ if $p_X = p_Y$.

2 Generalised Mehler Semigroups ([4], [9])

Let H be a real separable Hilbert space and $(S(t), t \geq 0)$ be a C_0-semigroup acting in H with infinitesimal generator J. Let $(\mu_t, t \geq 0))$ be a family of probability measures defined on $\mathcal{B}(H)$. Consider the space $C_b(H)$ of bounded continous real valued functions on H. It is a Banach space when equipped with the supremum norm. We define a family of bounded linear operators $(T(t), t \geq 0)$ on $C_b(H)$ by the prescription

$$(T(t)f)(x) = \int_H f(S(t)x + y)\mu_t(dy), \tag{2.1}$$

for each $t \geq 0, f \in C_b(H), x \in H$. It is shown in [4] that $(T(t), t \geq 0)$ is a semigroup if and only if $(\mu_t, t \geq 0)$ is a skew-convolution semigroup in the sense that

$$\mu_{t+r} = \mu_t * (\mu_r \circ (S(t)^{-1}) \tag{2.2}$$

for all $t, r \geq 0$.

If ν is a finite Borel measure on H we denote its Fourier transform by $\hat{\nu} : H \to \mathbb{C}$, so that

$$\hat{\nu}(y) = \int_H e^{i\langle x,y\rangle}\nu(dx),$$

for each $y \in H$. In [4] it is further established that if for all $y \in H$,

- the mapping $t \to \widehat{\mu_t}(y)$ is locally absolutely continuous on $[0, \infty)$ and differentiable at $t = 0$,
- the mapping $t \to \lambda(S(t)^*y)$ is locally Lebesgue integrable on $[0, \infty)$, where

$$\lambda(y) := -\frac{d}{dt}\widehat{\mu_t}(y)\bigg|_{t=0},$$

then (2.2) is equivalent to

$$\widehat{\mu_t}(v) = \exp\left\{-\int_0^t \lambda(S(r)^* v)dr\right\}, \tag{2.3}$$

for all $v \in H$. The mapping $\lambda : H \to \mathbb{C}$ is negative definite, hermitian and satisfies $\lambda(0) = 0$.

In the sequel we will always assume that the conditions given above hold. $(T(t), t \geq 0)$ is then called a (generalised) *Mehler semigroup.* We also assume that λ is Sazonov continuous, i.e. it is continuous with respect to the locally convex topology on H generated by the seminorms $y \to ||Ay||$, where A runs over all Hilbert-Schmidt operators in H. We can now utilise the Lévy-Khinchine formula in H (see e.g. Chapter 6 in [17]) to assert that λ must take the form

$$\begin{aligned} \lambda(y) &= -i\langle b, y\rangle + \frac{1}{2}\langle y, Qy\rangle \\ &+ \int_{H-\{0\}} (1 - e^{i\langle u,y\rangle} + i\langle u, y\rangle 1_{\{||u||<1\}})\nu(du), \end{aligned} \tag{2.4}$$

where $b \in H, Q$ is a positive, self-adjoint trace class operator on H and ν is a *Lévy measure* on $H - \{0\}$, i.e. $\int_{H-\{0\}}(||x||^2 \wedge 1)\nu(dx) < \infty$. The triple (b, Q, ν), which is the (set of) *characteristics* of λ, determines λ uniquely.

Note that $(T(t), t \geq 0)$ is not strongly continuous on $C_b(H)$ with the usual uniform topology. Provided $(\mu_t, t \geq 0)$ is weakly convergent to Dirac mass at the origin, it can be shown to be continuous with respect to the so-called mixed topology (see [10], [11]). We will not need such results in this paper.

3 Covariant Mehler Semigroups

Let G be a locally compact group and U be a continuous unitary (non-trivial) representation of G in H. For each $g \in G$, define $\mathcal{U}_g : C_b(H) \to C_b(H)$ by

$$(\mathcal{U}_g f)(x) = f(U_g x),$$

for each $f \in C_b(H), x \in H$, then $g \to \mathcal{U}_g$ is a anti-homomorphism of G into the automorphism group of the Banach algebra $C_b(H)$. We say that the Mehler semigroup $(T_t, t \geq 0)$ is covariant under the action $\mathcal{U}$ of G (or *G-covariant*, for short) if

$$T(t)\mathcal{U}_g = \mathcal{U}_g T(t),$$

for each $t \geq 0, g \in G$. In the sequel, $\rho^g := \rho \circ U_g^{-1}$, for all $g \in G$, whenever ρ is a function or measure defined on H.

Theorem 3.1 *The following are equivalent:*

(i) The Mehler semigroup $(T(t), t \geq 0)$ is G-covariant,

(ii) $\mu_t * \delta_{S(t)U_g x} = \delta_{U_g S(t)x} * \mu_t^g$,

(iii) $e^{i\langle v, S(t)U_g x\rangle}\widehat{\mu_t}(v) = e^{i\langle v, U_g S(t)x\rangle}\widehat{\mu_t^g}(v)$,

for all $g \in G, x, v \in H, t \geq 0$.

Proof. (i) $\Rightarrow$ (ii). For all $f \in C_b(H), x \in H$,

$$(\mathcal{U}_g T(t)f)(x) = \int_H f(S(t)U_g x + y)\mu_t(dy) = \int_H f(y)(\mu_t * \delta_{S(t)\mathcal{U}_g x})(dy).$$

$$\begin{aligned}(T(t)\mathcal{U}_g f)(x) &= \int_H f(U_g S(t)x + U_g y)\mu_t(dy) \\ &= \int_H f(U_g S(t)x + y)\mu_t^g(dy) \\ &= \int_h f(y)(\delta_{U_g S(t)x} * \mu_t^g)(dy).\end{aligned}$$

(ii) $\Rightarrow$ (i) is immediate.

(ii) $\Rightarrow$ (iii) Put $f(\cdot) = e^{i\langle v, \cdot\rangle}$.

(iii) $\Rightarrow$ (ii) follows from the fact that a finite measure is uniquely determined by its Fourier transform. □

It is clear from Theorem 3.1 that sufficient conditions for $(T(t), t \geq 0)$ to be G-covariant are

$$[S(t), U_g] = 0 \quad \text{and} \quad \mu_t^g = \mu_t, \tag{3.5}$$

for each $t \geq 0, g \in G$. To establish whether these are also necessary, we must probe at the infinitesimal level and from now on, we make the additional assumption that

$$U_g(\text{Dom}(J)) \subseteq \text{Dom}(J),$$

for all $g \in G$. Note that we can replace this by the weaker requirement that J has a U_g invariant core, for all $g \in G$, whenever such a core exists.

Theorem 3.2 *The following are equivalent:*

(i) The Mehler semigroup is G-covariant.

(iv) $U_g Jx = JU_g x$ *and* $\lambda^g = \lambda$*,*

(v) $[S(t), U_g] = 0$ *and* $\mu_t^g = \mu_t$*,*

for all $t \geq 0, g \in G, x \in Dom(J)$*.*

Proof. (i) ⇒ (iv) Apply (2.3) in Theorem 3.1 (iii) to obtain for each $t \geq 0, g \in G, x, v \in H$,

$$\exp\left\{-\int_0^t \{\lambda(S(r)^* U_g^{-1} v) - \lambda(S(r)^* v)\} dr\right\} = \exp\{i\langle v, [S(t), U_g]x\rangle\}.$$

Now take $x \in \text{Dom}(J)$, differentiate both sides of the above equation and let $t = 0$ to obtain

$$\lambda(v) - \lambda^g(v) = i\langle v, [J, U_g]x\rangle.$$

From this we deduce that the linear mapping from $\text{Dom}(J)$ to H given by $x \to [J, U_g]x$ is constant. But $0 \in \text{Dom}(J) \Rightarrow [J, U_g]x = 0$ for all $x \in \text{Dom}(J) \Rightarrow \lambda^g = \lambda$.

(iv) ⇒ (v) That $U_g Jx = JU_g x$ for all $x \in \text{Dom}(J) \Rightarrow [S(t), U_g] = 0$ is a standard (easily verified) fact about C_0-semigroups. If $\lambda^g = \lambda$ then by (2.3), we see that

$$\begin{aligned}
\widehat{\mu_t}^g(v) &= \exp\left\{-\int_0^t \lambda(S(r)^* U_g^{-1} v) dr\right\} \\
&= \exp\left\{-\int_0^t \lambda(U_g^{-1} S(r)^* v) dr\right\} \\
&= \exp\left\{-\int_0^t \lambda^g(S(r)^* v) dr\right\} \\
&= \widehat{\mu_t}(v).
\end{aligned}$$

But each $\widehat{\mu_t}^g = \widehat{\mu_t^g}$, hence $\mu_t^g = \mu_t$, by the uniqueness of Fourier transforms.

(v) ⇒ (i) is immediate □

It follows from the Lévy-Khintchine formula (2.4) that $\lambda^g = \lambda$ for all $g \in G$ if and only if

$$U_g b = b, \quad Q = U_g Q U_g^{-1}, \quad \nu^g = \nu, \tag{3.6}$$

for all $g \in G$. If (3.6) holds, we say that λ has *G-invariant characteristics.*

Corollary 3.1 *If G acts irreducibly on H, then $(T(t), t \geq 0)$ is G-covariant if and only if*

1. *There exists $\alpha \in \mathbb{R}$ such that $S(t) = e^{\alpha t} I$ for all $t \geq 0$,*

2. *λ has G-invariant characteristics with $b = 0, Q = cI$, for some $c \geq 0$.*

In the case where H is infinite-dimensional, (2) is replaced by
2′ λ has characteristics $(0, 0, \nu)$ where $\nu^g = \nu$, for all $g \in G$.

Proof. The condition (1) on the semigroup is a consequence of Schur's lemma. Using (3.6), we see that for G-covariance, the ray $\{\rho b, \rho \in \mathbb{R}\}$ must be invariant under the action of G, hence $b = 0$ by irreducibility when $\dim(H) \geq 2$. When $\dim(H) = 1, b = 0$ follows immediately from (3.6). From Schur's lemma again, we must have $Q = cI$ with $c \geq 0$, and in infinite dimensions Q cannot be trace class unless $c = 0$. □

We recall that the irreducible representations of compact Lie groups are always finite dimensional, so Corollary 3.1 makes most impact on noise reduction when G is non-compact.

4 Covariant Ornstein-Uhlenbeck Processes

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), P)$ be a stochastic base wherein the filtration $(\mathcal{F}_t, t \geq 0)$ satisfies the usual hypotheses of completeness and right continuity and let $X = (X(t), t \geq 0)$ be a (time homogeneous) Markov process defined on Ω and taking values in H. For each $t \geq 0, x \in H, A \in \mathcal{B}(H), p_t(x, A)$ is the transition probability $P(X(t) \in A | X(0) = x)$. We say that X has *covariant transitions* if for all $t \geq 0, x \in H, A \in \mathcal{B}(H), g \in G$:

$$p_t(U_g x, A) = p_t(x, U_g^{-1} A). \tag{4.7}$$

Let $(M(t), t \geq 0)$ be the transition semigroup of X acting on $B_b(H)$, so that $(M(t)f)(x) = \mathbb{E}(f(X(t)) | X(0) = x)$ for each $t \geq 0, x \in H, f \in B_b(H)$.

Proposition 4.1 $(M(t), t \geq 0)$ *is G-covariant if and only if X has G-covariant transition probabilities.*

Proof. This follows from the facts that for each $g \in G, x \in H, f \in B_b(H), t \geq 0$,

$$(\mathcal{U}_g M(t) f)(x) = \int_H f(y) p_t(U_g x, dy), \text{and}$$

$$(M(t)\mathcal{U}_g f)(x) = \int_H f(U_g y) p_t(x, dy) = \int_H f(y) p_t(x, U_g^{-1} dy).$$

□

We recall that a Markov process X is G-invariant if it has invariant laws, i.e. for all $t \geq 0, A \in \mathcal{B}(H), g \in G$,

$$p_{X(t)}(A) = p_{X(t)}(U_g^{-1}A).$$

For the next result we assume that X is normal, i.e. the mapping $x \to p_t(x, A)$ is measurable for each $t \geq 0, A \in \mathcal{B}(H)$.

Corollary 4.1 *If the normal Markov process X has G-invariant transition probabilities and the law of $X(0)$ is G-invariant, then X is G-invariant.*

Proof For each $t \geq 0, A \in \mathcal{B}(H), g \in G$,

$$\begin{aligned} p_{X(t)}(U_g^{-1}A) &= \int_H p_{X(0)}(dx)p_t(x, U_g^{-1}A) \\ &= \int_H p_{X(0)}(U_g dx)p_t(U_g x, A) \\ &= p_{X(t)}(A), \end{aligned}$$

where we have used the fact that each U_g is a bijection of H. □

Now let $Z = (Z(t), t \geq 0)$ be a Lévy process, i.e. a càdlàg, $\mathcal{F}_t$-adapted stochastically continuous process with stationary and independent increments for which $Z(0) = 0$ (a.s.). Then there exists a negative-definite, hermitian, Sazonov continuous function $\lambda : H \to \mathbb{C}$ with $\lambda(0) = 0$ such that

$$\mathbb{E}(e^{i\langle y, Z(t)\rangle}) = e^{-t\lambda(y)}, \tag{4.8}$$

for each $t \geq 0, y \in H$. Let (b, Q, ν) be the characteristics of λ.

The Lévy-Itô decomposition within this context has been established by Albeverio and Rüdiger ([1], see also [8]). It asserts that there exists a Brownian motion $(B_Q(t), t \geq 0)$ with covariance operator Q and an independent Poisson random measure N on $\mathbb{R}^+ \times (H - \{0\})$ with intensity measure $l \otimes \nu$ (where l is Lebesgue measure on $\mathbb{R}^+$) such that

$$Z(t) = tb + B_Q(t) + \int_{||x||<1} x\tilde{N}(t, dx) + \int_{||x||\geq 1} xN(t, dx), \tag{4.9}$$

where $\tilde{N}$ is the compensated Poisson measure, i.e. $\tilde{N}(dt, dx) = N(dt, dx) - dt\nu(dx)$.

Now consider the stochastic differential equation

$$dY(t) = JY(t)dt + dZ(t), \tag{4.10}$$

with initial condition $Y(0) = Y_0$, where Y_0 is $\mathcal{F}_0$-measurable and independent of $(Z(t), t \geq 0)$. This has been given a precise meaning using stochastic integration based on (4.9) in [2] and it is shown therein that it has a unique weak solution given by the *Ornstein-Uhlenbeck process*

$$Y(t) = S(t)Y(0) + \int_0^t S(t-r)dZ(r), \tag{4.11}$$

for each $t \geq 0$ (see [5] for an earlier alternative approach, and [4] for a method for obtaining strong solutions to (4.10) by extension to a larger Hilbert space). Y is a Markov process and its transition semigroup is a generalised Mehler semigroup and so has the form (2.1).

By Proposition 4.1, Y has covariant transition probabilities if and only if $(T(t), t \geq 0)$ is G-covariant. By Theorem 3.2, we see that this holds whenever J commutes with the group action and $\lambda^g = \lambda$, for all $g \in G$. By (4.8), the latter holds if and only if Z is G-invariant. If G acts irreducibly on H, then Y is a classical Ornstein-Uhlenbeck process driven by G-invariant Z, so that (4.10) becomes

$$dY(t) = \alpha Y(t)dt + dZ(t), \tag{4.12}$$

where $\alpha \in \mathbb{R}$.

Example Let $H = L^2(\mathbb{R}^d)$ and G be the orthogonal group $O(d)$. We consider the representation of G on $L^2(\mathbb{R}^d)$ given by $(L_g f)(x) = f(g^{-1}x)$, for each $g \in O(d), x \in \mathbb{R}^d$. Let $(S(t), t \geq 0)$ be the heat semigroup

$$(S(t)f)(x) = (2\pi t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(x+y)e^{-\frac{1}{2t}|y|^2}dy,$$

for each $t \geq 0, f \in H, x \in \mathbb{R}^d$. It is well-known (and easily checked) that $S(t)$ commutes with the group action. J is the usual Laplacian Δ and its domain is the Sobolev space

$$\mathcal{H}_2(\mathbb{R}^d) = \left\{ f \in H; \int_{\mathbb{R}^d} |y|^2 |\hat{f}(y)|^2 dy < \infty \right\},$$

where $\hat{f}$ is the Fourier transform of f, which is easily seen to be invariant under $O(d)$. The group action is, of course, reducible and the most general (left) $O(d)$-covariant Ornstein-Uhlenbeck process is that driven by a (left) $O(d)$-invariant Lévy process satisfying the conditions (3.6), e.g. we may take

$$dY(t) = \Delta Y(t)dt + dB_Q(t),$$

where Q is any positive, self-adjoint trace class operator in the commutant of the real von Neumann algebra $N = \{L_g, g \in G\}''$ (see Chapter 4 of [14] for a general account of such algebras). We may for example take $Q = S(1)$.

If G is a non-compact group acting irreducibly in G, then Y is of the form (4.12) with

$$Z(t) = \int_{||x||<1} x\tilde{N}(t,dx) + \int_{||x||\geq 1} xN(t,dx)$$

for each $t \geq 0$, and $\nu^g = \nu$ for each $g \in G$. It is easily verified that a sufficient condition for the latter is that

$$N(t,A) \stackrel{d}{=} N(t, U_g^{-1}A),$$

for each $t \geq 0, A \in \mathcal{B}(H), g \in G$. As an example, we take Z to be a G-invariant compound Poisson process,

$$Z(t) = X_1 + X_2 + \cdots + X_{N(t)},$$

where $(X_n, n \in \mathbb{N})$ are i.i.d random variables with common G-invariant law q and $(N(t), t \geq 0)$ is an independent Poisson process with intensity $h > 0$. In this case $\nu(\cdot) = hq(\cdot)$ is a finite measure and we can write the unique weak solution to (4.12) explicitly as

$$Y(t) = e^{\alpha t}Y_0 + \sum_{n\in\mathbb{N}} e^{\alpha(t-\tau_n)}X_n 1_{[0,t]}(\tau_n)$$

for each $t \geq 0$, where for each $n \in \mathbb{N}, \tau_n$ is the nth arrival time for $(N(t), t \geq 0)$ and has a gamma distribution with density $g_n(x) = \dfrac{h^{n-1}e^{-hx}x^{n-1}}{(n-1)!}1_{(0,\infty)}(x)$.

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