Brownian Motion and Lévy Processes in Locally Compact Groups

David Applebaum
Probability and Statistics Department,
University of Sheffield,
Hicks Building, Hounsfield Road,
Sheffield, England, S3 7RH *

e-mail: D.Applebaum@sheffield.ac.uk

Abstract

It is shown that every Lévy process on a locally compact group G is determined by a sequence of one-dimensional Brownian motions and an independent Poisson random measure. As a consequence, we are able to give a very straightforward proof of sample path continuity for Brownian motion in G. We also show that every Lévy process on G is of pure jump type, when G is totally disconnected.

MSC 2000: 60B15,60H20,60G44,22D05

1 Introduction

Let G be a separable locally compact group. A Lévy process on G is essentially a stochastic process with stationary and independent increments. In the case where $G = \mathbb{R}^d$, the study of these is a classical area of investigation for probability theory which continues to yield a wealth of interesting results. Two recent monographs on this subject are [11] and [27]. A key to the evolution of the theory at this level is the celebrated Lévy-Khintchine formula, which characterises Lévy processes by means of their characteristic functions.

In the case where G is a Lie group, the mathematical development begins with a beautiful paper by G.A.Hunt [21] which classifies the infinitesimal generators of the associated Markov semigroup. This result is equivalent to the Lévy-Khintchine formula in Euclidean space. More recently, it was shown that the paths of the process were determined by a Brownian motion in Euclidean space and an independent Poisson random measure on the group [3]. For a recent survey of results in the Lie group case, see [5].

*Most of this work was carried out while the author was employed by the Nottingham Trent University

Now consider the case where G is an arbitrary locally compact group. For abelian G, Lévy processes (or equivalently, weakly continuous convolution semigroups of probability measures) were investigated in the 1960s. In this case, a Fourier transform can be built with the aid of the dual group, and a Lévy-Khinchine formula is obtained which directly generalises the Euclidean space case. A good source for this is Chapter IV of Parthasarathy [25].

In the non-abelian case, probabilistic investigations were greatly assisted by discoveries made during the 1950's, in the course of trying to solve Hilbert's 5th problem, which demonstrated that any locally compact group contained a projective limit of Lie groups as an open neighborhood of the identity [18]. In his seminal monograph [20] (Chapter IV), Heyer was able to generalise Hunt's result in the Lie group case, to show that the generator of a Lévy process is a sum of three mappings, which can be interpreted probabilistically as describing drift, diffusion and jumps, respectively.

Recent developments which have maintained interest include;

1. Born [12] has shown that any projective limit of Lie algebras has a projective basis. This enabled him [13] to rewrite Heyer's formula for the generator, in a form where it is transparently an infinite dimensional generalisation of Hunt's result in the Lie group case. This new formulation is much more amenable to probabilistic investigations. We remark that Heyer and Pap [19] have recently extended Born's result to study the generation of two parameter semigroups of probability measures, which correspond to additive processes, i.e. those having independent, but not necessarily stationary, increments.

2. In a series of recent papers, Bendikov and Saloff-Coste have obtained important new results about Brownian motion on compact groups, see e.g. [9] for studies of sample path regularity, and [8] for investigations of the case where the law of the process is absolutely continuous with respect to Haar measure, and has a continuous density. These results have interesting implications for the potential theory of associated harmonic sheaves.

There are a number of interesting examples of locally compact groups which are not Lie groups. These include the infinite-dimensional torus and the p-adic solenoid. A nice account of both of these, which is in a pertinent form for the present article, can be found in section 6 of [19].

The organisation of this paper is as follows. In the first section, we gather together all the results we need on projective limits of Lie groups and Lie algebras and describe the key results of Born on projective bases [12]. In the second part, we show that the results of [3] described above for Lie groups, pass over directly to the locally compact case. So any Lévy process on G is determined by a Euclidean Brownian motion, coupled to the Lie algebra through a given projective basis, and an independent Poisson random measure which lives on the group.

We then discuss Brownian motion on G, which is defined to be a Lévy process which is Gaussian (see [20], Chapter VI, section 2) . We give a short probabilistic proof of sample path continuity, Although this result has been proved in a more general setting by Siebert [29], the proof presented here has the advantage of brevity. Finally we generalise a result due to Evans [16] in the abelian case, whereby every Lévy process in a totally disconnected group is shown to be of pure jump type.

Notation. All topological groups discussed in this article, will be assumed to be Hausdorff. If G is a topological group, then G_0 will denote the connected component of the identity e. $\mathcal{B}(G)$ is the σ-algebra of Borel subsets of G. If G is a locally compact group, $B_b(G)$ and $C_0(G)$ are the Banach spaces (when equipped with the supremum norm) of bounded Borel functions and continuous functions which vanish at infinity, respectively. For each $\tau \in G, L_\tau$ denotes left translation, which is the isometric isomorphism of $B_b(G)$ given by $(L_\tau f)(\sigma) = f(\tau\sigma)$, for each $f \in B_b(G), \sigma \in G$. $\mathcal{K}(G)$ is the norm dense subalgebra of $C_0(G)$ comprising smooth functions of compact support. If I is an index set and $(f_i, i \in I)$ is a family of real-valued functions on G, we write $f : G \to \mathbb{R}$ as $f = \bigoplus_{i\in I} f_i$ whenever

1. The domains $(\mathrm{Dom}(f_i), i \in I)$ form a partition of G.
2. $f(\sigma) = f_i(\sigma)$ for all $\sigma \in \mathrm{Dom}(f_i)$.
3. $f_i = 0$ for all but a finite number of $i \in I$.

If G is a Lie group, then its Lie algebra will sometimes be denoted by $\mathcal{L}(G)$.

Acknowledgements. I would like to thank Herbert Heyer for drawing my attention to Born's papers, and Sasha Bendikov for helpful discussions.

2 Projective limits and the Lie algebra of a locally compact group

Let $(I, <)$ be a partially ordered set. Suppose that for every $i \in I$, there exists a locally compact group G_i, such that for every $i, j \in I$ with $i < j$, there is a continuous open homomorphism $\pi_{ij} : G_j \to G_i$, such that $\pi_{ik} = \pi_{ij} \circ \pi_{jk}$, for all $i < j < k$. The *projective limit* $\varprojlim_{i\in I} G_i$ is the closed subgroup $\left\{(x_i, i \in I) \in \prod_{i\in I} G_i; x_i = \pi_{ij}(x_j) \text{ for all } i, j \in I, i < j\right\}$.
Yamabe [30] has proved that every connected locally compact group can be represented as a projective limit of Lie groups.
Let $(H_i, i \in I)$ be a family of compact, normal subgroups of a locally compact group G. We say that they form a *Lie system* if

1. $i < j \Rightarrow H_j \subseteq H_i$.
2. $\bigcap_{i\in I} H_i = \{e\}$.
3. G/H_i is a Lie group for all $i \in I$.

A locally compact group G is said to be *Lie projective* if there exists a Lie system $(H_i, i \in I)$ such that $G = \varprojlim_{i\in I} G/H_i$. Gluškov [18] has proved that in every locally

compact group G there exists an open Lie projective subgroup $G_1 \supseteq G_0$. Note that the Lie system so associated to G_1 is not necessarily unique.
We now turn our attention to Lie algebras. A *topological Lie algebra* is a (not necessarily finite dimensional) Lie algebra for which the Lie bracket is jointly continuous in the vector topology. Projective limits of Lie algebras were introduced by Lashoff [24]. Suppose that for every $i \in I$, there exists a topological Lie algebra $\mathbf{g}_i$, such that for every $i, j \in I$ with $i < j$, there is a continuous open Lie algebra homomorphism $p_{ij} : \mathbf{g}_j \to \mathbf{g}_i$, such that $p_{ik} = p_{ij} \circ p_{jk}$, for all $i < j < k$. The *projective limit* $\varprojlim_{i\in I} \mathbf{g}_i$ is the closed subalgebra $\left\{(X_i, i \in I) \in \prod_{i\in I} \mathbf{g}_i; X_i = p_{ij}(X_j) \text{ for all } i, j \in I, i < j\right\}$.
The relationship between projective Lie groups and projective Lie algebras is straightforward when $G = \varprojlim_{i\in I} G_i$, with each G_i a Lie group. In this case $\mathbf{g} = \varprojlim_{i\in I} \mathcal{L}(G_i)$ is a topological Lie algebra wherein $p_{ij} = d\pi_{ij}$, for each $i, j \in I, i < j$. We then call $\mathbf{g}$ the *Lie algebra* of the locally compact group G and sometimes denote it by $\mathcal{L}(G)$. There is a natural notion of exponential map from $\mathcal{L}(G)$ to G which works as follows. If $X = (X_i, i \in I) \in \mathcal{L}(G)$, then

$$\exp(X) = (\exp(X_i), i \in I).$$

For each $X \in \mathcal{L}(G)$, the map $t \to \exp(tX)$ is a continuous homomorphism from $\mathbb{R}$ to G. We define the *left invariant vector field* X^L associated to X in the obvious way, i.e.

$$(X^L f)(\sigma) = \lim_{h\to 0} \frac{f(\sigma \exp(hX)) - f(\sigma)}{h},$$

where $f \in C(G)$ is such that the limit on the right hand side exists for all $\sigma \in G$. We say that f is uniformly differentiable with respect to $X \in \mathcal{L}(G)$ if the above limit exists in the supremum norm, for all $f \in C_0(G)$.
If G is an arbitrary locally compact group, we can apply Gluškov's theorem to define the Lie algebra $\mathcal{L}(G)$ of G to be that of G_1, so that

$$\mathcal{L}(G) = \varprojlim_{i\in I} \mathcal{L}\left(G_1/H_i\right).$$

From now on, we will always work within this context.
For each $i \in I, \pi_i$ is the canonical surjection from G_1 onto G_1/H_j and $d\pi_i$ is then the canonical surjection from $\mathcal{L}(G)$ onto $\mathcal{L}(G_1/H_j)$.
Born [12] introduced the following important concept:-
Let S be a set for which $I \subseteq S$. A family $(X_i, i \in S)$ in $\mathcal{L}(G) - \{0\}$ is called a *projective basis* if for each $j \in I$, there is a finite subset $S_j \subset S$, such that $(d\pi_j(X_i), i \in S_j)$ is a basis for $\mathcal{L}(G/H_j)$ and $d\pi_j(X_i) = 0$ whenever $i \notin S_j$.

Theorem 2.1 (Born) *If G is a locally compact group and G_1 is a Lie projective open subgroup of G, then there exists a Lie system for G_1 with respect to which $\mathcal{L}(G)$ has a projective basis $(X_i, i \in S)$.*

The proof can be found in [12].

We need the following spaces of differentiable functions which were introduced by Born [13]. For each $k \in \mathbb{N}, j \in I$, let $C_k(G/H_j)$ be the linear subspace of $C_0(G/H_j)$ comprising functions which are k-times uniformly differentiable with respect to k-fold products from the set $(d\pi_j(X_i), i \in S_j)$. We define

$$C_k(G_1) = \bigcup_{j \in I} \{f^j \circ \pi_j, f^j \in C_k(G/H_j)\},$$

$$\text{and } C_k(G) = \bigoplus_{z \in Z} \{L_z f, f \in C_k(G_1)\},$$

where Z generates a representative set of left cosets of G_1.
For each $k \in \mathbb{N}, C_k(G)$ is norm dense in $C_0(G)$. The space $\mathcal{D}(G)$ of infinitely differentiable (with respect to the given projective basis) functions of compact support on G is constructed in a similar manner to the above (see also [14]). $\mathcal{D}(G)_+$ will denote the non-negative elements of $\mathcal{D}(G)$.
A *weak co-ordinate system* ([13]) for G at e (relative to a given projective basis $(X_i, i \in S)$) is a family $(k_i, i \in S)$, such that $k_i \in \mathcal{D}(G), (X_i k_j)(e) = \delta_{ij}$ and $k_i(\sigma) = -k_i(\sigma^{-1})$, for each $i, j \in S, \sigma \in G$.

Theorem 2.2 (Born) *Let G be a locally compact group and G_1 be a Lie projective open subgroup of G. Suppose that there exists a projective basis $(X_i, i \in S)$ for $\mathcal{L}(G)$ corresponding to some Lie system of G_1. Then there exists a weak co-ordinate system relative to $(X_i, i \in S)$.*

See [13] for the proof.

We will need a final result from [12]. Let $\mathcal{F}(S)$ be the set of all finite subsets of S. It is a directed set by inclusion.

Proposition 2.1 *If $(t_i, i \in S)$ with each $t_i \in \mathbb{R}$, there is exactly one $X \in \mathcal{L}(G)$ for which*

$$X = \lim_{J \in \mathcal{F}(S)} \sum_{j \in J} t_j X_j.$$

If $X \in \mathcal{L}(G)$ is as in Proposition 2.1, we write $X = \sum_{i \in S} t_i X_i$. The associated left-invariant vector field is $X^L = \sum_{i \in S} t_i X_i^L$. We can similarly define second-order differential operators of the form $A = \sum_{i,j \in S} a_{ij} X_i^L X_j^L$, where each $a_{ij} \in \mathbb{R}$.

3 Lévy Processes in Locally Compact Groups

Let G be a locally compact separable group and let $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), P)$ be a stochastic base, so that $(\Omega, \mathcal{F}, P)$ is a probability space and $(\mathcal{F}_t, t \geq 0)$ is a filtration of σ-algebras satisfying the standard conditions of right-continuity and completeness. $\phi = (\phi(t), t \geq 0)$ will denote a $\mathcal{F}_t$-adapted process defined on $(\Omega, \mathcal{F}, P)$, and taking values in G.
We say that ϕ is a *Lévy process in G* if it satisfies the following:

1. ϕ has stationary and independent left increments
2. $\phi(0) = e$ (a.s.)
3. ϕ is stochastically continuous, i.e.

$$\lim_{s \to t} P(\phi(s)^{-1}\phi(t) \in A) = 0$$

for all $A \in \mathcal{B}(G)$ with $e \notin \bar{A}$.

Now let $(p_t, t \geq 0)$ be the law of the Lévy process ϕ, then it follows from the definition that $(p_t, t \geq 0)$ is a weakly continuous convolution semigroup of probability measures on G, where the convolution operation is defined for measures μ and ν on G by

$$(\mu * \nu)(A) = \int_G \mu(d\tau)\nu(\tau^{-1}A),$$

for each $A \in \mathcal{B}(G)$. So that in particular we have, for all $s, t \geq 0$

$$p_{s+t} = p_s * p_t \quad \text{and} \quad \text{wklim}_{t \to 0} p_t = \delta_e, \tag{3.1}$$

where δ_e is Dirac measure concentrated at e. Since ϕ is a Markov process, we obtain a contraction semigroup of linear operators $(T(t), t \geq 0)$ on $B_b(G)$ by the prescription

$$(T(t)f)(\tau) = \mathbb{E}(f(\tau\phi(t))) = \int_G f(\tau\sigma)p_t(d\sigma), \tag{3.2}$$

for each $t \geq 0, f \in C_0(G), \tau \in G$. In fact $(T(t), t \geq 0)$ is a Feller semigroup in that

$$T(t)(C_0(G)) \subseteq C_0(G) \;\text{ and }\; \lim_{t \to 0} ||T(t)f - f|| = 0$$

for each $f \in C_0(G)$.
Note that $L_\tau T(t) = T(t)L_\tau$, for each $\tau \in G, t \geq 0$. Let $\mathcal{A} : \text{Dom}(\mathcal{A}) \to C_0(G)$ be the infinitesimal generator of $(T(t), t \geq 0)$. In order to describe $\mathcal{A}$ explicitly, we need the notion of Lévy measure ([20], p.296). This is a measure ν on $(G - \{e\}, \mathcal{B}(G - \{e\}))$ for which

1. $\nu(U^c) < \infty$ for every neighbourhood of the identity U in G.
2. $\int_{G-\{e\}} f(\sigma)\nu(d\sigma) < \infty$ for all $f \in \mathcal{D}(G)_+$ with $f(e) = 0$.

We have the following key result for the structure of $\mathcal{A}$. The explicit form given below in terms of a projective basis is due to Born [13]. This is based on an earlier result by Heyer [20] pp. 300-8, who expressed the right hand side of (3.3) below as a sum of three different types of mapping.

Theorem 3.1 *Let G be a locally compact group with fixed projective basis $(X_i, i \in S)$ and weak co-ordinate system $(k_i, i \in S)$. If $\phi = (\phi(t), t \geq 0)$ is a Lévy process in G, then $C_2(G) \subseteq Dom(\mathcal{A})$ and for all $f \in C_2(G), \sigma \in G$,*

$$\begin{aligned} \mathcal{A}f(\sigma) &= \sum_{i\in S} b^i X_i^L f(\sigma) + \sum_{i,j\in S} a^{ij} X_i^L X_j^L f(\sigma) \\ &+ \int_{G-\{e\}} (f(\sigma\tau) - f(\sigma) - \sum_{i\in S} k_i(\tau) X_i^L f(\sigma))\nu(d\tau) \end{aligned} \quad (3.3)$$

where $b = (b_i, i \in S) \in \mathbb{R}^S$, $a = (a_{ij}, i, j \in S)$ is a non-negative symmetric matrix and ν is a Lévy measure on $G - \{e\}$.

The triple (b, a, ν) is called the *characteristics* of the Lévy process ϕ.
We remark that when we evaluate these at a specific $f \in C_2(G)$, each of the sums in (3.3) becomes finite.
It was shown by Heyer [20], p.308 that the Lévy measure ν is uniquely determined by the convolution semigroup $(p_t, t \geq 0)$ by means of

$$\int_{G-\{e\}} f(\sigma)\nu(d\sigma) = \lim_{t\to 0} \frac{1}{t} \int_G f(\sigma)p_t(d\sigma), \quad (3.4)$$

for all $f \in \mathcal{K}(G - \{e\})$.
In the sequel, we will without further comment, utilise the specific projective basis and associated weak co-ordinate system of Theorem 3.1.
From now on we will assume that the group G has a countable basis for its topology. It then follows that G is metrizable (see e.g. [23], Chapter 4) and we can find a countable Lie system $(H_n, n \in \mathbb{N})$ in G_1. We say that a Lévy process is càdlàg if almost all of it paths are right continuous with left limits. In this case, we write $\Delta\phi(t) = \phi(t-)^{-1}\phi(t)$ for each $t \geq 0$, where $\phi(t-)$ is the left limit. Note that $t \to \Delta\phi(t)$ takes at most countably many non-zero values on compact intervals.
The main result of this section is the following theorem. The proof is essentially the same as the Lie group case which was established in [3] and so we will just sketch the main steps rather than repeating the full argument. In the following, we take $\mathcal{F}_t = \sigma\{\phi(s), 0 \leq s \leq t\}$, for each $t \geq 0$.

Theorem 3.2 *If $\phi = (\phi(t), t \geq 0)$ is a càdlàg Lévy process in G with infinitesimal generator $\mathcal{A}$ of the form (3.3), then there exists*

- *an $\mathcal{F}_t$-adapted Poisson random measure N on $\mathbb{R}^+ \times (G - \{e\})$,*
- *a sequence $B = (B_n, n \in \mathbb{N})$, which is independent of N, of one-dimensional $\mathcal{F}_t$-adapted Brownian motions with mean zero and covariance $Cov(B_n(t), B_m(t) = 2ta_{mn}$, for each $t \geq 0, m, n \in \mathbb{N}$,*

such that for each $f \in C_2(G), t \geq 0$,

$$\begin{aligned} f(\phi(t)) &= f(e) + \sum_{n\in\mathbb{N}} \int_0^t (X_n^L f)(\phi(s-))dB^n(s) + \int_0^t \mathcal{A}f(\phi(s-))ds + \\ &+ \int_0^{t+} \int_{G-\{e\}} (f(\phi(s-)\tau) - f(\phi(s-))\tilde{N}(ds, d\tau), \end{aligned} \tag{3.5}$$

where $\tilde{N}(ds, d\tau) = N(ds, d\tau) - ds\nu(d\tau)$.
Furthermore, ϕ is uniquely determined by B and N and

$$\mathcal{F}_t = \sigma\{B_n(s), N((s,t] \times E); n \in \mathbb{N}, 0 \leq s \leq t, E \in \mathcal{B}(G - \{e\})\},$$

for each $t \geq 0$.

Proof (Sketch). For each $0 \leq s \leq t < \infty, \sigma \in G$, we introduce the notation $\phi_{s,t}(\sigma) = \sigma\phi(s)^{-1}\phi(t)$. Now fix $s \geq 0$. For each $f \in C_2(G), \sigma \in G$, define $M_s^{f,\sigma} = (M_{s,t}^{f,\sigma}, t \geq s)$ by

$$M_{s,t}^{f,\sigma} = f(\phi_{s,t}(\sigma)) - f(e) - \int_s^t \mathcal{A}f(\phi_{s,u}(\sigma))du.$$

Then $M_s^{f,\sigma}$ is a centred L^2-martingale. We can compute the associated Meyer angle bracket to obtain

$$\langle M_{s,t}^{f,\sigma_1}, M_{s,t}^{g,\sigma_2}\rangle = \int_s^t B(f,g)(\phi_{s,u}(\sigma_1), \phi_{s,u}(\sigma_2))du,$$

for each $f, g \in C_2(G), \sigma_1, \sigma_2 \in G$, where the “carré de champ”

$$B(f,g)(\rho_1,\rho_2) = 2\sum_{m,n=1}^{\infty} a_{m,n}(X_m^L f)(\rho_1)(X_n^L g)(\rho_2) + \int_{G-\{e\}} (f(\rho_1\tau) - f(\rho_1))(g(\rho_2\tau) - g(\rho_2))\nu(d\tau),$$

for each $\rho_1, \rho_2 \in G$. Now let $\mathcal{P} = \{0 = t_0 < t_1 < t_2 < \cdots\}$ be a partition of $\mathbb{R}^+$ with mesh $\delta(\mathcal{P}) = \max_{n\in\mathbb{N}}(t_n - t_{n-1}) < \infty$. We define a centred L^2-martingale $(Y_t^{\mathcal{P},f,\sigma}, t \geq 0)$ by

$$Y_t^{\mathcal{P},f,\sigma} = \sum_{n\in\mathbb{N}} M_{t\wedge t_{n-1}, t\wedge t_n}^{f,\sigma},$$

for each $t \geq 0$. Then we obtain another centred L^2-martingale $(Y_t^{f,\sigma}, t \geq 0)$ by

$$Y_t^{f,\sigma} = \mathrm{L}^2 - \lim_{\delta(\mathcal{P})\to 0} Y_t^{\mathcal{P},f,\sigma},$$

for each $t \geq 0$. Moreover, for each $f, g \in C_2(G), \sigma_1, \sigma_2 \in G, t \geq 0$, we have

$$\langle Y_t^{f,\sigma_1}, Y_t^{g,\sigma_2}\rangle = tB(f,g)(\sigma_1,\sigma_2) \text{ and } M_{s,t}^{f,\sigma} = \int_s^t dY_u^{f,\phi_{s,u-}(\sigma)},$$

in the sense of the non-linear stochastic integral of Fujiwara and Kunita [17], lemma 4.2.
For each $t \geq 0$, let $Y_t^{f,\sigma} = Y_t^{(c),f,\sigma} + Y_t^{(d),f,\sigma}$ be the unique decomposition into continuous and discontinuous centred martingales. For each $0 \leq s \leq t < \infty, E \in \mathcal{B}(G - \{e\})$, define

$$N((s,t], E) = \#\{0 \leq s < u \leq t; \Delta\phi(u) \in E\}.$$

Then N extends to a Poisson random measure on $\mathbb{R}^+ \times (G - \{e\})$ with intensity measure ν, and for each $t \geq 0$,

$$Y_t^{(d),f,\sigma} = \int_0^{t+} \int_{G-\{e\}} (f(\sigma\tau) - f(\sigma)\tilde{N}(ds, d\tau).$$

For each $n \in \mathbb{N}, t \geq 0$, define $B_n(t) = Y_t^{(c),k_n,e}$. Then for each $m, n \in \mathbb{N}$, $\langle B_m(t), B_n(t)\rangle = 2ta_{m,n}$. Hence for each $n \in \mathbb{N}, t \geq 0, (B_n(t), t \geq 0)$ is a Brownian motion, by Lévy's characterisation, and for each $t \geq 0$,

$$Y_t^{(c),f,\sigma} = \sum_{n \in N} X_n^L f(\sigma) B^n(t),$$

from which the required result follows. $\square$

Example: The Compound Poisson Process

Let $(X_n, n \in \mathbb{N})$ be a sequence of i.i.d. random variables taking values in G, with common law q, and let $(M(t), t \geq 0)$ be a (non-negative-integer-valued) Poisson process, with intensity $\lambda > 0$, which is independent of all the X_n's. We define the *compound Poisson process* by

$$\phi(0) = e; \quad \phi(t) = X_1 X_2 \cdots X_{N(t)} \text{ for } t > 0.$$

Elementary calculations, show that (3.5) can be written as:

$$f(\phi(t)) = f(e) + \int_0^{t+} \int_{G-\{e\}} (f(\phi(s-)\tau) - f(\phi(s-)))N(ds, d\tau),$$

for all $f \in C_0(G), t \geq 0$. We then find that $\mathcal{A}$ is a bounded linear operator in $C_0(G)$ with

$$\mathcal{A}f(\sigma) = \int_{G-\{e\}} (f(\sigma\tau) - f(\sigma))\nu(d\tau),$$

for all $f \in C_0(G), \sigma \in G$, where $\nu(\cdot) = \lambda q(\cdot)$. See [4] for details.

A consequence of Theorem 3.2 is the strong Markov property for the Lévy process ϕ. Let T be an $\mathcal{F}_t$-stopping time and for each $t \geq 0$, define the process $\phi_T = (\phi_T(t), t \geq 0)$ by

$$\phi_T(t) = \phi(T)^{-1}\phi(T + t).$$

Let $\mathcal{F}_T$ denote the stopped σ-algebra so that

$$\mathcal{F}_T = \sigma\{A \in \mathcal{F}; A \cap \{T \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0\}.$$

Theorem 3.3 (Strong Markov Property) *If ϕ is a càdlàg Lévy process in G, then ϕ_T is a càdlàg Lévy process on G, which is independent of $\mathcal{F}_T$ and has the same characteristics as ϕ.*

Proof. Once we have Theorem 3.2, the proof proceeds along the same lines as that of the case where G is a Lie group. We refer to theorem 2 of [4] for the details. $\square$

4 Brownian Motion In Locally Compact Groups

Let ϕ be a càdlàg Lévy process in G. We say that ϕ is a *Brownian motion* if the associated convolution semigroup of laws $(p_t, t \geq 0)$ is *Gaussian*, i.e.

$$\lim_{t \to 0} \frac{1}{t} p_t(U^c) = 0,$$

for every neighbourhood U of e in G.

Note. This definition is due to Siebert [28] (based on considerations by Courrège [15] section 8, in the Euclidean case). It has been extensively developed by Heyer [20], section 6.2.

Theorem 4.1 *If ϕ is a Lévy process in G with characteristics (b, a, ν), then ϕ is a Brownian motion in G if and only if $\nu = 0$ and $a \neq 0$.*

Proof. This is due to Heyer [20], theorem 6.2.20, pp. 440 - 1. $\square$

By Theorem 4.1, we see that a càdlàg Lévy process is a Brownian motion in G if and only if the action of its infinitesimal generator on $C_2(G)$ is given by

$$\mathcal{A} = \sum_{m,n \in \mathbb{N}} a_{mn} X_m^L X_n^L + \sum_{n \in \mathbb{N}} b_n X_n^L,$$

with the matrix $a \neq 0$.

Corollary 4.1 *If ϕ is a càdlàg Lévy process in G, then ϕ is a Brownian motion if and only if there exists a sequence $B = (B_n, n \in \mathbb{N})$ of one-dimensional $\mathcal{F}_t$-adapted Brownian motions with mean zero and non-zero covariance $Cov(B_n(t), B_m(t) = 2ta_{mn}$, for each $t \geq 0, m, n \in \mathbb{N}$, such that for each $f \in C_2(G), t \geq 0$,*

$$\begin{aligned} f(\phi(t)) &= f(e) + \sum_{n \in \mathbb{N}} \int_0^t (X_n^L f)(\phi(s)) dB^n(s) + \sum_{m,n \in \mathbb{N}} \int_0^t a_{mn} X_m^L X_n^L f(\phi(s)) ds \\ &+ \sum_{n \in \mathbb{N}} \int_0^t b_n X_n^L f(\phi_n(s)) ds. \end{aligned} \tag{4.6}$$

Proof This follows immediately from Theorems 3.2 and 4.1 . $\square$

If $a = 0$, in (4.6), then $\phi(t) = \exp\left(t\sum_{n\in\mathbb{N}} b_n X_n\right)$, for each $t \geq 0$. We call such a deterministic process a *pure drift*.

Theorem 4.2 *If ϕ is a Lévy process, then the sample paths of ϕ are a.s. continuous if and only if ϕ is a Brownian motion or a pure drift.*

Proof. Let N be the Poisson random measure associated to ϕ. Since $\mathbb{E}(N(t,A)) = t\nu(A)$, for all $t \geq 0, A \in \mathcal{B}(G - \{e\})$, it follows by Theorem 4.1 that ϕ is a Brownian motion or a pure drift if and only if $N = 0$ (a.s.). By the construction in Theorem 3.2, this holds if and only if $\Delta\phi(t) = 0$ (a.s.), for each $t \geq 0$ and the result follows. $\square$

Note. For a proof that Gaussianity or pure drift and (a.s.) path-continuity are equivalent in a more general setting, see Siebert [29]. In the symmetric case (see below), a.s. continuity of Brownian motion in G may also be proved by checking that the semigroup acting in $L^2(G,m)$ has the property $\langle T(t)u, v\rangle = o(t)$, as $t \to 0$, for all continuous functions u and v having disjoint compact supports (see Bendikov [10]). In [6], pp. 1210 -11, it is pointed out that this holds whenever the generator is a local operator, which is certainly true in this case.

The following result exhibits the projective structure of the Brownian motion ϕ. Let $\tau = \inf\{t \geq 0; \phi(t) \notin G_1\}$.

Theorem 4.3 *For each $0 \leq t < \tau, n \in \mathbb{N}, (\pi_n(\phi(t)), t \geq 0)$ is a Brownian motion in the Lie group G/H_n.*

Proof. For each $n \in \mathbb{N}$, let $d_n = \dim(G/H_n)$ and let $X_{i_1}^{(n)}, \ldots, X_{i_{d_n}}^{(n)}$ be the basis of $\mathcal{L}(G/H_n)$ given by $X_{i_j}^{(n)} = d\pi_n(X_j)$, for each $j \in S_n$. Let $f \in C_2(G)$ be of the form $f = f_n \circ \pi_n$, where $f_n \in C_2(G/H_n)$. Write $\phi_n(t) = \pi_n(\phi(t))$, for each $n \in \mathbb{N}, t \geq 0$. By Corollary 4.1, we obtain

$$
\begin{aligned}
f_n(\phi_n(t)) &= f_n(e) + \sum_{j=i_1^{(n)}}^{i_{d_n}^{(n)}} \int_0^t (X_j^L f_n)(\phi_n(s)) dB^n(s) \\
&+ \sum_{j,k=i_1^{(n)}}^{i_{d_n}^{(n)}} \int_0^t a_{jk} X_j^L X_k^L f_n(\phi_n(s)) ds + \sum_{j=i_1^{(n)}}^{i_{d_n}^{(n)}} \int_0^t b_j X_j^L f_n(\phi_n(s)) ds.
\end{aligned}
$$

But then we have that ϕ_n is the solution of the Stratonovitch stochastic differential equation

$$
d\phi_n(t) = \sum_{j=i_1^{(n)}}^{i_{d_n}^{(n)}} X_j^L(\phi_n(t)) \circ dB^n(t) + \sum_{j=i_1^{(n)}}^{i_{d_n}^{(n)}} b_j X_j^L(\phi_n(t)),
$$

with initial condition $\phi_n(0) = e$.
Hence ϕ_n is a (left-invariant) Brownian motion on G/H_n (see Itô [22]), as required. $\square$

Of course, $\tau = \infty$ (a.s.) if G is Lie projective, i.e. $G = G_1$.

Note. In the proof of Theorem 4.3, we showed each projection of Brownian motion onto the Lie group G/H_n satisfies a Stratonovitch stochastic differential equation. Readers may speculate if it is possible to represent ϕ itself as a solution of an equation of the form:

$$\text{“}d\phi(t) = \sum_{n\in\mathbb{N}} X_n^L(\phi(t)) \circ dB^n(t) + \sum_{n\in\mathbb{N}} b_n X_n^L(\phi(t))\text{”}.$$

In fact, this could be carried out using techniques applied by Albeverio and Daletskii to the construction of left-invariant diffusions on infinite products of Lie groups [1, 2]; however this requires that the X_n's satisfy a number of constraints that seem unnatural within the present context.

A Brownian motion ϕ is said to be *symmetric* if

$$p_t(A) = p_t(A^{-1}),$$

for all $A \in \mathcal{B}(G)$. The following result is well-known (see e.g. [7], p.1282). We indicate a proof for completeness

Theorem 4.4 *If a Brownian motion is symmetric then the action of its infinitesimal generator on $C_2(G)$ is given by*

$$\mathcal{A} = \sum_{m,n\in\mathbb{N}} a_{mn} X_m^L X_n^L.$$

Proof. (Sketch). Let $(T(t), t \geq 0)$ be the Markov semigroup acting in $C_0(G)$. Restrict to $\mathcal{D}(G)$ and then extend to a Markov semigroup acting in $L^2(G, m)$, where m is a right-invariant Haar measure. Then $(T(t), t \geq 0)$ is easily seen to be self-adjoint and so has a self-adjoint generator. Hence $b = 0$. $\square$

5 Lévy Processes in Totally Disconnected Groups

In this section we take G to be totally disconnected. Every Lévy process in G is then of “pure jump” type as the following theorem shows:-

Theorem 5.1 *If ϕ is a Lévy process in a totally disconnected locally compact group G, then there exists a Poisson random measure N on $\mathbb{R}^+ \times (G - \{e\})$ such that for all $f \in C_2(G), t \geq 0$,*

1. *if $k_n \notin L^1(G - \{e\}, \nu)$ for all $n \in \mathbb{N}$*

$$f(\phi(t)) = f(e) + \int_0^{t+} \int_{G-\{e\}} (f(\phi(s-)\tau) - f(\phi(s-))\tilde{N}(ds, d\tau) \tag{5.7}$$

$$+ \quad \int_0^{t+} \int_{G-\{e\}} [f(\phi(s-)\tau) - f(\phi(s-)) - \sum_{n\in\mathbb{N}} k_n(\tau) X_n^L f(\phi(s-))]\nu(d\tau)ds;$$

2. if $k_n \in L^1(G - \{e\}, \nu)$ for all $n \in \mathbb{N}$,

$$f(\phi(t)) = f(e) + \int_0^{t+} \int_{G-\{e\}} (f(\phi(s-)\tau) - f(\phi(s-))N(ds, d\tau). \tag{5.8}$$

Proof. Let ϕ be a Lévy process with characteristics (b, a, ν). We assume, without loss of generality, that the support of ν is the whole of $G - \{e\}$. Let $(U_n, n \in \mathbb{N})$ be a sequence of neighborhoods of the identity in G for which $U_1 \subset G$ and $U_n \downarrow \{e\}$ as $n \to \infty$. We define a sequence of (finite) Lévy measures $(\nu_n, n \in \mathbb{N})$ by the prescription

$$\nu_n(A) = \nu(A \cap U_n^c),$$

for all $A \in \mathcal{B}(G - \{e\})$. Let ϕ_n be a Lévy process with characteristics (b, a, ν_n). For each $r \in \mathbb{N}$, let $(T_k^{(r)}, k \in \mathbb{N})$ be the jump times of the Poisson process $(N(t, U_r^c), t \geq 0)$. For each $T_k^{(r)} \leq t < T_{k+1}^{(r)}$, (3.5) yields

$$\begin{aligned}
f(\phi_r(t)) &= f(\phi_r(T_k^{(r)})) + \sum_{n\in\mathbb{N}} \int_{T_k^{(r)}}^t (X_n^L f)(\phi_r(s-))dB^n(s) + \int_{T_k^{(r)}}^t \mathcal{A}f(\phi_r(s-)ds \\
&- \int_{T_k^{(r)}}^t \int_{U_r^c} (f(\phi_r(s-)\tau) - f(\phi_r(s-))\nu(d\tau)ds \\
&= f(\phi_r(T_k^{(r)})) + \sum_{n\in\mathbb{N}} \int_{T_k^{(r)}}^t (X_n^L f)(\phi_r(s-))dB^n(s) \\
&+ \int_{T_k^{(r)}}^t \left\{ \sum_{n\in\mathbb{N}} \left(b_n - \int_{U_r^c} k_n(\tau)\nu(d\tau) \right) X_n^L f(\phi_r(s-)) \right. \\
&+ \left. \sum_{m,n\in\mathbb{N}} a_{mn} X_m^L X_n^L f(\phi_r(s-)) \right\} ds.
\end{aligned}$$

By Corollary 4.1 and Theorem 4.2, it follows that ϕ_r has continuous sample paths for all $T_k^{(r)} \leq t < T_{k+1}^{(r)}$. But G is totally disconnected and so ϕ_r is constant on each $[T_k^{(r)}, T_{k+1}^{(r)})$. Hence

$$f(\phi_r(t)) = f(e) + \int_0^{t+} \int_{U_r^c} (f(\phi_r(s-)\tau) - f(\phi_r(s-))N(ds, d\tau),$$

and consequently $a = 0$ and $b_n - \int_{U_r^c} k_n(\tau)\nu(d\tau) = 0$, for all $r, n \in \mathbb{N}$. But then we must have $b_n = \int_{U_1^c} k_n(\tau)\nu(d\tau)$ and $\int_{U_r^c \cap U_1} k_n(\tau)\nu(d\tau) = 0$, for each $r, n \in \mathbb{N}$. Hence if $k_n \notin L^1(G - \{e\}, \nu)$, we deduce (5.7) immediately from (3.5). If $k_n \in L^1(G - \{e\}, \nu)$, we have

$$\int_{U_1 - \{e\}} k_n(\tau)\nu(d\tau) = \lim_{r\to\infty} \int_{U_r^c \cap U_1} k_n(\tau)\nu(d\tau) = 0,$$

for all $n \in \mathbb{N}$, and (5.8) follows. $\square$

In fact, it is not difficult to verify that if ϕ satisfies (5.8) then it is of the form

$$\phi(t) = \prod_{0 \leq s \leq t} \Delta\phi(s),$$

for each $t \geq 0$, where the product is time-ordered from left to right.

Notes

- A similar result to that of Theorem 5.1 was established by Evans [16] in the case where G is abelian.
- If G is discrete, then every Lévy process on G is of the form (5.8). Its generator $\mathcal{A}$ then takes the form

$$(\mathcal{A}f)(\sigma) = \int_G (f(\sigma\tau) - f(\sigma))\nu(d\tau),$$

for $f \in C_2(G), \sigma \in G$. This representation was obtained by Ramaswami [26] using different techniques and it was further shown that ν is finite, so that $\mathcal{A}$ extends to a bounded operator on $C_0(G)$ and ϕ is a compound Poisson process.

References

[1] S.Albeverio, A.Daletskii, Stochastic equations and quasi-invariance on infinite product groups, *Inf. Diml. Anal. Quant. Prob. and Rel. Topics* **2**, 283-8 (1999)

[2] S.Albeverio, A.Daletskii, Quasi-invariance and Gibbs structure of diffusion measures on infinite product groups, *Methods of Functional Analysis and Topology*, **6**, 28-42 (2000)

[3] D.Applebaum, H. Kunita, Lévy flows on manifolds and Lévy processes on Lie groups, *J.Math Kyoto Univ* **33**, 1103-23 (1993)

[4] D.Applebaum, Compound Poisson processes and Lévy processes in groups and symmetric spaces, *J.Theor. Prob.* **13**, 383-425 (2000)

[5] D.Applebaum, Lévy processes in stochastic differential geometry in *Lévy Processes: Theory and Applications* ed. O.Barndorff-Nielsen, T.Mikosch, S.Resnick (Birkhäuser Boston Basel Berlin), 111-39 (2001)

[6] A.Bendikov, L.Saloff-Coste, On- and off-diagonal heat kernel behaviours on certain infinite-dimensional local Dirichlet spaces, *Amer. J. Math.* **122**, 1205-63 (2000)

[7] A.Bendikov, L.Saloff-Coste, Gaussian bounds for derivatives of central Gaussian semigroups on compact groups, *Trans. Amer. Math. Soc.* **354**, 1279-98 (2001)

[8] A.Bendikov, L.Saloff-Coste, Central Gaussian semigroups of measures with continuous density, *J. Funct. Anal.* **186**, 206-86 (2001)

[9] A.Bendikov, L.Saloff-Coste, On the sample paths of Brownian motions on compact infinite dimensional groups, *Annals of Prob.*, **31**, 1464-94 (2003)

[10] A.Bendikov, A continuity criterion for one class of Markov processes, *Theor. Prob. Appl.* **21**, 169-71 (1976)

[11] J.Bertoin, *Lévy Processes*, Cambridge University Press (1996)

[12] E.Born, Projective Lie algebra bases of a locally compact group and uniform differentiability, *Math Z.* **200**, 279-92 (1989)

[13] E.Born, An explicit Lévy-Hinčin formula for convolution semigroups on locally compact groups, *J.Theor. Prob.* **2** 325-42 (1989)

[14] F.Bruhat, Distributions sur un groupe localement compact et applications à l'étude des représentations des groupes p-adiques, *Bull. Soc. Math. France* **89**, 43-75 (1961)

[15] P.Courrège, Génerateur infinitésimal d'un semi-group de convolution sur $\mathbb{R}^n$ et formule de Lévy-Khintchine, *Bull. Sci. Math. 2e Série* **88**, 3-30 (1964)

[16] S.N.Evans, Local properties of Lévy processes on a totally disconnected group, *J.Theor. Prob.* **2** 209-59 (1989)

[17] T.Fujiwara, H.Kunita, Stochastic differential equations of jump type and Lévy processes in diffeomorphisms group, *J.Math.Kyoto Univ* **25**, 71-106 (1985)

[18] V.M.Gluškov, The structure of locally compact groups and Hilbert's fifth problem, *American Mathematical Society Translations* Ser.**15**, 55-93 (1957)

[19] H.Heyer, G.Pap, Convolution hemigroups of bounded variation on a Lie projective group, *J. London Math. Soc.* **59**, 369-84 (1999)

[20] H.Heyer, *Probability Measures on Locally Compact Groups*, Springer-Verlag, Berlin-Heidelberg (1977)

[21] G.A.Hunt, Semigroups of measures on Lie groups, *Trans. Amer. Math. Soc.* **81**, 264-93 (1956)

[22] K.Itô, Brownian motions on a Lie group, *Proc. Jap. Acad.* **26**, 4-10 (1950)

[23] J.L.Kelley *General Topology*, Springer-Verlag, New York-Berlin-Heidelberg-Tokyo (1955)

[24] R.K.Lashoff, Lie algebras of locally compact groups, *Pacific J.Math.* **7**, 1145-62 (1957)

[25] K.R.Parthasarthy, *Probability Measures on Metric Spaces*, Academic Press, New York (1967)

[26] S.Ramaswami, Semigroups of measures on Lie groups *J. Indian Math. Soc.* **38**, 175-89 (1974)

[27] K-I.Sato, *Lévy Processes and Infinite Divisibility*, Cambridge University Press (1999)

[28] E.Siebert, Stetige Halbgruppen von Wahrscheinlichkeitsmaβen auf localkompacten maximal fastperiodischen Gruppen, *Z.Wahrscheinlichkeitstheorie verw. Gebiete* **25**, 269-300 (1973)

[29] E.Siebert, Jumps of stochastic processes with values in a topological group, *Probab. Math. Stat.* **5**, 197-209 (1985)

[30] H.Yamabe, A generalisation of a theorem of Gleason, *Annals of Math.* **58**, 351-65 (1953)