MAS350 solutions (2016-17)

1. (i) We have $P(\emptyset) = cP_1(\emptyset) + (1-c)P_2(\emptyset) = 0$, $P(S) = cP_1(S) + (1-c)P_2(S) = 1$ and for $A_1, A_2, \cdots$ disjoint

$$P(\cup_{n=1}^{\infty} A_n) = cP_1(\cup_{n=1}^{\infty} A_n) + (1-c)P_2(\cup_{n=1}^{\infty} A_n)$$
$$= c\sum_{n=1}^{\infty} P_1(A_n) + (1-c)\sum_{n=1}^{\infty} P_2(A_n) = \sum_{n=1}^{\infty} P(A_n).$$

(ii) (a) Consider the disjoint sets $C_1 = A_1$, $C_2 = A_2 - A_1$, $C_3 = A_3 - A_2 \cdots$. Then $A_N = \cup_{n=1}^{N} C_n$. Therefore

$$m(A) = \sum_{n=1}^{\infty} m(C_n) = \lim_{N\to\infty} \sum_{n=1}^{N} m(C_n) = \lim_{N\to\infty} m(A_N).$$

(b) The sets $A_n = S - B_n$ are increasing. Therefore

$$m(B) = m(\cap_{n=1}^{\infty} B_n) = m(\cap_{n=1}^{\infty} A_n^c) = m\big((\cup_{n=1}^{\infty} A_n)^c\big)$$
$$= m(S) - m(\cup_{n=1}^{\infty} A_n) = m(S) - \lim_{n\to\infty} m(A_n) = \lim_{n\to\infty} m(B_n)$$

(c) Consider $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ and the sets $B_n = [n, \infty)$. Note that $\cap_{n=1}^{\infty}[n,\infty) = \emptyset$ but $m(B_n) = \infty$.

(iii) Σ should contain $A \cap B = \{3\}$, $A - B = \{1,2\}$ and $B - A = \{4,5\}$. Take unions of these sets

$$\{\emptyset, \{1,2\}, \{3\}, \{4,5\}, \{1,2,3\}, \{1,2,4,5\}, \{3,4,5\}, S\}$$

One can check that this is a σ-algebra. Any σ-algebra that contains A and B must contain the above sets.

(iv) This follows from

$$[a,b] = \cap_{n=1}^{\infty}(a - n^{-1}, b + n^{-1})$$

and

$$\{a\} = \cap_{n=1}^{\infty}(a - n^{-1}, a + n^{-1})$$

2. (i) (a) $\limsup_{n\to\infty} a_n = 1$ and $\liminf_{n\to\infty} a_n = -1$

(b) Yes, the limit is -1

(ii) (a) Just observe that

$$(f + \mathbf{1})^{-1}([a,\infty)) = \{s \in S : f(s) + 1 \in [a,\infty)\} = f^{-1}([a-1,\infty)) \in \Sigma$$

since f is measurable.

(b) Let $r_n, n \geq 1$ be an enumeration of the rationals.

$$\{f > g\} = \cup_{n=1}^{\infty}\{f > r_n > g\} = \cup_{n=1}^{\infty}(\{f > r_n\} \cap \{g < r_n\}) \in \Sigma.$$

This is because both $\{f > r_n\}$ and $\{g < r_n\}$ are in Σ by the measurability of f and g.

(iii) $\mathbf{1}_{A\cap B} = 1$ if and only if $x \in A \cap B$. When $x \in A \cap B$ both $\mathbf{1}_A(x)$ and $\mathbf{1}_B(x)$ are 1. When $x \notin A \cap B$ at least one of $\mathbf{1}_A(x)$ and $\mathbf{1}_B(x)$ is 0. This proves the identity.

(iv) A simple example is $S = \{0,1\}$, $\Sigma = \{\emptyset, S\}$. Let $f : S \to \mathbb{R}$ be such that $f(0) = -1$ and $f(1) = 1$. We see that $|f|$ is a constant function and therefore measurable with respect to Σ whereas f is not measurable since $f^{-1}(\{-1\}) = \{0\}$ which is not in Σ.

(v) If f is constant on both A and A^c then it is of the form $f = c_1\mathbf{1}_A + c_2\mathbf{1}_{A^c}$. This is a simple function and is therefore measurable.

Next take any measurable f. We must have $f^{-1}(\{a\}) \in \Sigma$ for all $a \in \mathbb{R}$. Take $a \in f(A)$ and $b \in f(A^c)$. The set $f^{-1}(\{a\})$ is in Σ and should be either A or S. Similarly $f^{-1}(\{b\})$ is either A^c or S. In any case this implies that f has to be constant on both A and A^c.

3. (i) (a) One can see

$$f_+ = \begin{cases} 1, & 0 \le x < 1 \\ 2, & 10 \le x < 11, \\ 0 & \text{otherwise} \end{cases}, \qquad f_- = \begin{cases} 2, & 1 \le x < 2 \\ 3 & 2 \le x < 4, \\ 0 & \text{otherwise.} \end{cases}$$

(b) To compute the integral we first compute the integrals of f_+ and f_-. Both are simple functions and thus are easily computed

$$\int f_+(x)dx = 3 \quad \int f_-(x)dx = 8$$

Therefore $\int f(x)dx = -5$.

(ii) (a) Write

$$f = f \cdot \mathbf{1}_{f=0} + f \cdot \mathbf{1}_{f \neq 0} = f \cdot \mathbf{1}_{f \neq 0}.$$

Let $A = \{f \neq 0\}$.

$$\int f dm = \int_A f_+ dm - \int_A f_- dm = 0$$

since $m(A) = 0$.

(b) We know that $\lambda(\mathbf{1}_{\mathbb{Q}} \neq 0) = \lambda(\mathbb{Q}) = 0$ and therefore the integral is 0

(c) It is false. Consider $([0,1], \mathcal{B}([0,1]), \lambda)$ with $f(x) = 1$ if $x < 1/2$ and $f(x) = -1$ if $x \geq 1/2$. This function has integral 0 but is nonzero almost everywhere.

(iii) For a nonnegative function f we have

$$\int f dm = \sup\{s : s \text{ simple function with } s \le f\}.$$

Since $f \le g$ we have a bigger collection for g over which we take the supremum. Thus the integral of g should be larger than the integral of f.

(iv) We use Markov's inequality as follows

$$\begin{aligned} \lambda\left(x \in [0,2] : f(x) \ge 4\right) &= \lambda\left(x \in (1,2] : f(x) \ge 4\right) \\ &\le \frac{\int_1^2 f(x)dx}{4} \le 1/2 \end{aligned}$$

4. (i) (a) Let (S, Σ, m) be a measure space. If (f_n) is a sequence of non-negative measurable functions from S to $\mathbb{R}$ then

$$\liminf_{n\to\infty} \int_S f_n dm \ge \int_S \liminf_{n\to\infty} f_n dm$$

(b) Fatou's lemma gives

$$\liminf_{n\to\infty} \int_S (f - f_n)dm \ge \int_S \liminf_{n\to\infty} (f - f_n)dm$$

Therefore by linearity of the integral

$$\liminf_{n\to\infty} \int_S f dm + \liminf_{n\to\infty} - \int_S f_n dm \ge \int_S \liminf_{n\to\infty} f dm + \int_S \liminf_{n\to\infty} -f_n dm$$

We know that $\liminf_{n\to\infty} -a_n = -\limsup_{n\to\infty} a_n$ for any sequence a_n and so we get

$$\limsup_{n\to\infty} \int f_n dm \le \int \limsup_{n\to\infty} f_n dm.$$

(ii) (a) Since $|(1 - e^{-|x|})f(x)| \le |f(x)|$ we can conclude that $(1 - e^{-|x|})f(x)$ is integrable.

(b) Dominated convergence theorem: Let (S, Σ, m) be a measure space. Let (f_n) be a sequence of measurable functions from S to $\mathbb{R}$ which converges pointwise to a (measurable) function f. Suppose there is an integrable function $g : S \to \mathbb{R}R$ so that $|f_n| \leq g$ for all $n \in \mathbb{N}$. Then f is integrable and

$$\int_S f dm = \lim_{n\to\infty} \int_S f_n dm.$$

With $f_n = (1 - e^{-|x|/n})e^{-|x|}$, we see that $\lim_{n\to\infty} f_n(x) = e^{-|x|}$ for all x. Further $|f_n| \leq e^{-|x|}$ for all n and $e^{-|x|}$ is an integrable function. Therefore

$$\lim_{n\to\infty} \int (1 - e^{-|x|/n}) f(x) dx = \int e^{-|x|} dx = 2.$$

(iii) Consider $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ and $f_n = \mathbf{1}_{[n,\infty)}$. The functions f_n decrease to the function $f \equiv 0$. We have $\int f_n d\lambda = \infty$ for all n but $\int f d\lambda = 0$.

(iv) (a) By the triangle inequality $|f(x) + g(x)| \leq |f(x)| + |g(x)|$ for all x. Therefore

$$\int |f(x) + g(x)| dx \leq \int |f(x)| + |g(x)| dx = \int |f(x)| dx + \int |g(x)| dx.$$

(b) We use the above

$$\int |f_n + g_n - (f + g)| dm \leq \int |f_n - f| dm + \int |g_n - g| dm.$$

Since both the terms on the right hand side go to 0 so does the left hand side.

5. (i) (a) For ω such that $\{X(\omega) = k\}$, the left hand side is equal to k. The right hand side

$$\sum_{i=1}^{\infty} \mathbf{1}_{\{X(\omega)\geq i} = \sum_{i=1}^{k} \mathbf{1}_{\{X(\omega)\geq i\}} = k.$$

Note that the disjoint events $\{X(\omega) = k\}$ cover the entire space.

(b) We use the above along with the monotone convergence theorem

$$\mathbb{E}(X) = \mathbb{E}\Big[\sum_{i=1}^{\infty} \mathbf{1}_{\{X\geq i\}}\Big] = \sum_{i=1}^{\infty} \mathbb{E}[1_{\{X\geq i\}}] = \sum_{i=1}^{\infty} \mathbb{P}(X \geq i)$$

(ii) Consider the events

$A_k = \{$The five consecutive tosses starting from the $(5k + 1)$th toss are $HTHHT\}$

Clearly the events $A_k, k \geq 1$ are independent because they depend on independent coin tosses. Further $P(A_k) = 2^{-5}$ and so $\sum_{k=1}^{\infty} P(A_k) = \infty$. Therefore by the Borel Cantelli lemma we have $P(A_n$ occurs i.o. $) = 1$.

(iii) We compute

$$E(e^{itX}) = \sum_{k=0}^{\infty} e^{-2} \frac{e^{itk} 2^k}{k!} = e^{-2} \exp(2e^{it}) = \exp\big(2(e^{it} - 1)\big),\ t \in \mathbb{R}$$

(iv) Since f, g are Borel measurable we have for $A, B \in \mathcal{B}(\mathbb{R})$ that $f^{-1}(A),\ g^{-1}(B) \in \mathcal{B}(\mathbb{R})$. Therefore

$$P(f(X) \in A,\ g(Y) \in B) = P(X \in f^{-1}(A)) \cdot P(Y \in g^{-1}(B)) = P(f(X) \in A) \cdot P(g(Y) \in B)$$

which implies the independence of $f(X)$ and $g(Y)$.

(v) To show that X_n converges a.s. to 0 observe that $X_n(0) = 0$ and for any $a > 0$ we have $X_n(a) = 0$ for all $n \geq 1/a$. We also see

$$\mathbb{E}(X_n^2) = n^2 \cdot \frac{1}{n} = n$$

which does not converge to 0 and so X_n does not converge in mean square to 0.