MAS451/6352 solutions (2016-17)

1. (i) We have $P(\emptyset) = cP_1(\emptyset) + (1-c)P_2(\emptyset) = 0$, $P(S) = cP_1(S) + (1-c)P_2(S) = 1$ and for $A_1, A_2, \cdots$ disjoint

$$\begin{aligned} P(\cup_{n=1}^{\infty} A_n) &= cP_1(\cup_{n=1}^{\infty} A_n) + (1-c)P_2(\cup_{n=1}^{\infty} A_n) \\ &= c\sum_{n=1}^{\infty} P_1(A_n) + (1-c)\sum_{n=1}^{\infty} P_2(A_n) = \sum_{n=1}^{\infty} P(A_n). \end{aligned}$$

(ii) (a) Consider the disjoint sets $C_1 = A_1$, $C_2 = A_2 - A_1$, $C_3 = A_3 - A_2 \cdots$. Then $A_N = \cup_{n=1}^{N} C_n$. Therefore

$$m(A) = \sum_{n=1}^{\infty} m(C_n) = \lim_{N\to\infty} \sum_{n=1}^{N} m(C_n) = \lim_{N\to\infty} m(A_N).$$

(b) The sets $A_n = S - B_n$ are increasing. Therefore

$$\begin{aligned} m(B) = m(\cap_{n=1}^{\infty} B_n) &= m(\cap_{n=1}^{\infty} A_n^c) = m\big((\cup_{n=1}^{\infty} A_n)^c\big) \\ &= m(S) - m(\cup_{n=1}^{\infty} A_n) = m(S) - \lim_{n\to\infty} m(A_n) = \lim_{n\to\infty} m(B_n) \end{aligned}$$

(c) Consider $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ and the sets $B_n = [n, \infty)$. Note that $\cap_{n=1}^{\infty}[n,\infty) = \emptyset$ but $m(B_n) = \infty$.

(iii) Σ should contain $A \cap B = \{3\}$, $A - B = \{1,2\}$ and $B - A = \{4,5\}$. Take unions of these sets

$$\{\emptyset, \{1,2\}, \{3\}, \{4,5\}, \{1,2,3\}, \{1,2,4,5\}, \{3,4,5\}, S\}$$

One can check that this is a σ-algebra. Any σ-algebra that contains A and B must contain the above sets.

(iv) This follows from

$$[a,b] = \cap_{n=1}^{\infty}(a - n^{-1}, b + n^{-1})$$

and

$$\{a\} = \cap_{n=1}^{\infty}(a - n^{-1}, a + n^{-1})$$

(v) (a) We first check that $\mathcal{C}$ is a σ-algebra.

- Clearly $\emptyset \in \mathcal{C}$.
- We show that $\mathcal{C}$ is closed under complements. Suppose that $A \in \mathcal{C}$. Since $A \in \mathcal{B}(\mathbb{R})$ we have $A^c \in \mathcal{B}(\mathbb{R})$. We just need to show $-A^c \in \mathcal{B}(\mathbb{R})$ and for this it is enough to show that $-A^c = (-A)^c$, since $-A \in \mathcal{B}(\mathbb{R})$. To see this

$$x \in -A^c \iff -x \in A^c \iff -x \notin A \iff x \notin -A \iff x \in (-A)^c$$

- Next suppose that $A_1, A_2, \cdots \in \mathcal{C}$. We have that $\cup_{i=1}^{\infty} A_i \in \mathcal{B}(\mathbb{R})$. We now only need to show that $-\cup_{i=1}^{\infty} A_i \in \mathcal{B}(\mathbb{R})$. For this it is enough to prove that $-\cup_{i=1}^{\infty} A_i = \cup_{i=1}^{\infty}(-A_i)$. This follows from

$$\begin{aligned} &x \in -\cup_{i=1}^{\infty} A_i \iff -x \in \cup_{i=1}^{\infty} A_i \iff -x \in A_i \text{ for some } i \\ &\iff x \in -A_i \text{ for some } i \iff x \in \cup_{i=1}^{\infty}(-A_i) \end{aligned}$$

(b) Note $(a,b) \in \mathcal{C}$ for any $-\infty \le a < b \le \infty$ because $-(a,b) = (-b,-a)$. Since $\mathcal{B}(\mathbb{R})$ is the smallest σ-algebra containing the intervals we must have have $\mathcal{B}(\mathbb{R}) \subseteq C$. But by definition $\mathcal{C} \subseteq \mathcal{B}(\mathbb{R})$. Thus equality holds.

2. (i) (a) $\limsup_{n\to\infty} a_n = 1$ and $\liminf_{n\to\infty} a_n = -1$

(b) Yes, the limit is -1

(ii) (a) Just observe that

$$(f+\mathbf{1})^{-1}([a,\infty)) = \{s \in S : f(s)+1 \in [a,\infty)\} = f^{-1}([a-1,\infty)) \in \Sigma$$

since f is measurable.

(b) Let $r_n, n \geq 1$ be an enumeration of the rationals.

$$\{f > g\} = \cup_{n=1}^{\infty}\{f > r_n > g\} = \cup_{n=1}^{\infty}(\{f > r_n\} \cap \{g < r_n\}) \in \Sigma.$$

This is because both $\{f > r_n\}$ and $\{g < r_n\}$ are in Σ by the measurability of f and g.

(iii) A simple example is $S = \{0,1\}$, $\Sigma = \{\emptyset, S\}$. Let $f : S \to \mathbb{R}$ be such that $f(0) = -1$ and $f(1) = 1$. We see that $|f|$ is a constant function and therefore measurable with respect to Σ whereas f is not measurable since $f^{-1}(\{-1\}) = \{0\}$ which is not in Σ.

(iv) If f is constant on both A and A^c then it is of the form $f = c_1\mathbf{1}_A + c_2\mathbf{1}_{A^c}$. This is a simple function and is therefore measurable.

Next take any measurable f. We must have $f^{-1}(\{a\}) \in \Sigma$ for all $a \in \mathbb{R}$. Take $a \in A$ and $b \in A^c$. The set $f^{-1}(\{a\})$ is in Σ and should be either A or S. Similarly $f^{-1}(\{b\})$ is either A^c or S. In any case this implies that f has to be constant on both A and A^c.

(v) (a) Write

$$f = f \cdot \mathbf{1}_{f=0} + f \cdot \mathbf{1}_{f\neq 0} = f \cdot \mathbf{1}_{f\neq 0}.$$

Let $A = \{f \neq 0\}$.

$$\int f dm = \int_A f_+ dm - \int_A f_- dm = 0$$

since $m(A) = 0$.

(b) We know that $\lambda(\mathbf{1}_{\mathbb{Q}} \neq 0) = \lambda(\mathbb{Q}) = 0$ and therefore the integral is 0.

(c) It is false. Consider $([0,1], \mathcal{B}([0,1]), \lambda)$ with $f(x) = 1$ if $x < 1/2$ and $f(x) = -1$ if $x \geq 1/2$. This function has integral 0 but is nonzero almost everywhere.

(vi) We use Markov's inequality as follows

$$\begin{aligned}\lambda\left(x \in [0,2] : f(x) \geq 4\right) &= \lambda\left(x \in (1,2] : f(x) \geq 4\right) \\ &\leq \frac{\int_1^2 f(x)dx}{4} \leq 1/2\end{aligned}$$

3. (i) Dominated convergence theorem: Let (S, Σ, m) be a measure space. Let (f_n) be a sequence of measurable functions from S to $\mathbb{R}$ which converges pointwise to a (measurable) function f. Suppose there is an integrable function $g : S \to \mathbb{R}R$ so that $|f_n| \leq g$ for all $n \in \mathbb{N}$. Then f is integrable and

$$\int_S f dm = \lim_{n\to\infty} \int_S f_n dm.$$

With $f_n = (1 - e^{-|x|/n})e^{-|x|}$, we see that $\lim_{n\to\infty} f_n(x) = e^{-|x|}$ for all x. Further $|f_n| \leq e^{-|x|}$ for all n and $e^{-|x|}$ is an integrable function. Therefore

$$\lim_{n\to\infty} \int (1 - e^{-|x|/n}) f(x) dx = \int e^{-|x|} dx = 2.$$

(ii) Consider $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ and $f_n = \mathbf{1}_{[n,\infty)}$. The functions f_n decrease to the function $f \equiv 0$. We have $\int f_n d\lambda = \infty$ for all n but $\int f d\lambda = 0$.

(iii) Fatou's lemma gives

$$\liminf_{n\to\infty} \int_S (f - f_n) dm \geq \int_S \liminf_{n\to\infty} (f - f_n) dm$$

Therefore by linearity of the integral

$$\liminf_{n\to\infty} \int_S f dm + \liminf_{n\to\infty} -\int_S f_n dm \geq \int_S \liminf_{n\to\infty} f dm + \int_S \liminf_{n\to\infty} -f_n dm$$

We know that $\liminf_{n\to\infty} -a_n = -\limsup_{n\to\infty} a_n$ for any sequence a_n and so we get

$$\limsup_{n\to\infty} \int f_n dm \leq \int \limsup_{n\to\infty} f_n dm.$$

(iv) (a) For ω such that $\{X(\omega) = k\}$, the left hand side is equal to k. The right hand side

$$\sum_{i=1}^{\infty} \mathbf{1}_{\{X(\omega)\geq i} = \sum_{i=1}^{k} \mathbf{1}_{\{X(\omega)\geq i\}} = k.$$

Note that the disjoint events $\{X(\omega) = k\}$ cover the entire space.

(b) We use the above along with the monotone convergence theorem

$$\mathbb{E}(X) = \mathbb{E}\Big[\sum_{i=1}^{\infty} \mathbf{1}_{\{X\geq i\}}\Big] = \sum_{i=1}^{\infty} \mathbb{E}[1_{\{X\geq i\}}] = \sum_{i=1}^{\infty} \mathbb{P}(X \geq i)$$

(v) Consider the events

$$A_k = \{\text{The five consecutive tosses starting from the } (5k+1)\text{th toss are } HTHHT\}$$

Clearly the events $A_k, k \geq 1$ are independent because they depend on independent coin tosses. Further $P(A_k) = 2^{-5}$ and so $\sum_{k=1}^{\infty} P(A_k) = \infty$. Therefore by the Borel Cantelli lemma we have $P(A_n$ occurs i.o. $) = 1$.

(vi) To show that X_n converges a.s. to 0 observe that $X_n(0) = 0$ and for any $a > 0$ we have $X_n(a) = 0$ for all $n \geq 1/a$. We also see

$$\mathbb{E}(X_n^2) = n^2 \cdot \frac{1}{n} = n$$

which does not converge to 0 and so X_n does not converge in mean square to 0.

(vii) We have

$$\mathbb{E}\big[(S_{n+1}^2 - (n+1)) \cdot \mathbf{1}_{A_n}\big] = \mathbb{E}\big[\{S_n^2 + X_{n+1}^2 - 2X_{n+1}S_n - (n+1)\} \cdot \mathbf{1}_{A_n}\big]$$

First note that

$$E(\{X_{n+1}^2 - 1\} \cdot \mathbf{1}_{A_n}) = E(X_{n+1}^2 - 1) \cdot P(A_n) = 0.$$

This is because X_{n+1} is independent of $\mathbf{1}_{A_n}$ since it is a function of S_n. Similarly

$$E(X_{n+1} \cdot S_n \mathbf{1}_{A_n}) = E(X_{n+1}) \cdot E(S_n \mathbf{1}_{A_n}) = 0.$$

This proves the result.

4. (i) (a) This follows from

$$y \in (E\cap F)_x \iff (x,y) \in E\cap F \iff (x,y) \in E \text{ and } (x,y) \in F \iff y \in E_x \text{ and } y \in F_x$$

(b) This follows from

$$y \in (E^c)_x \iff (x,y) \in E^c \iff (x,y) \notin E \iff y \notin E_x \iff y \in (E_x)^c$$

(c) This follows from

$$y \in (\cup_{n=1}^{\infty} E_n)_x \iff (x,y) \in \cup_{n=1}^{\infty} E_n \iff (x,y) \in E_n \text{ for some } n$$
$$\iff y \in (E_n)_x \text{ for some } n \iff y \in \cup_{n=1}^{\infty} (E_n)_x$$

(ii) Let $f : S_1 \times S_2 \to \mathbb{R}$ be a non-negative measurable function. Then the mappings

$$x \to \int_{S_2} f(x,y) dm_2(y),$$

$$\text{and } y \to \int_{S_1} f(x,y) dm_1(x),$$

are both measurable. Furthermore

$$\begin{aligned}\int_{S_1 \times S_2} f d(m_1 \times m_2) &= \int_{S_1} \left(\int_{S_2} f(x,y) dm_2(y) \right) dm_1(x) \\ &= \int_{S_2} \left(\int_{S_1} f(x,y) dm_1(x) \right) dm_2(y).\end{aligned}$$

(iii) (a) Let r_n be an enumeration of the rationals. Note

$$\begin{aligned}G^c &= \{(\omega, y) : X(\omega) < y\} \\ &= \cup_{n=1}^{\infty} \{(\omega, y) : X(\omega) < r_n < y\} \\ &= \cup_{n=1}^{\infty} \big(\{X(\omega) < r_n\} \cap \{r_n < y\} \big)\end{aligned}$$

Both the sets $\{(\omega, y) : X(\omega) < r_n\}$ and $\{(\omega, y) : r_n < y\}$ are in the product σ-algebra $\mathcal{F} \otimes \mathcal{B}([0,1])$. Therefore it is clear that G^c and hence G is in $\mathcal{F} \otimes \mathcal{B}([0,1])$.

(b) We use Fubini's theorem to deduce

$$\begin{aligned}P \times \lambda(G) &= \int_{\Omega \times [0,1]} \mathbf{1}_G \, d(P \times \lambda) \\ &= \int_{\Omega} \Big(\int_{[0,1]} \mathbf{1}_G d\lambda \Big) dP \\ &= \int_{\Omega} \lambda([0, X(\omega)]) dP \\ &= \int_{\Omega} X(\omega) \, dP = \mathbb{E}X\end{aligned}$$

(iv) (a) Let us check the conditions for the collection $\mathcal{C}$ is a λ system

- $S \in \mathcal{C}$ because $m_1(S) = m_2(S)$.
- Let E_n be an increasing sequence of sets in $\mathcal{C}$. Therefore $m_1(E_n) = m_2(E_n)$. We know $\lim_{n\to\infty} m_1(E_n) = m_1(\cup_{n=1}^{\infty} E_n)$ and similarly for m_2. Therefore $m_1(\cup_{n=1}^{\infty} E_n) = m_2(\cup_{n=1}^{\infty} E_n)$ and so $\cup_{n=1}^{\infty} E_n \in \mathcal{C}$.
- Let $E \subseteq F \in \mathcal{C}$. Then $m_1(F - E) = m_1(F) - m_1(E) = m_2(F) - m_2(E) = m_2(F - E)$ and so $F - E \in \mathcal{C}$.

(b) Let m be a measure satisfying $m(J) = \lambda(J)$ for all subintervals of $[0,1]$. Consider the collection

$$\mathcal{D} = \{A \in \mathcal{B}([0,1]) : m(A) = \lambda(A)\}$$

From the previous step $\mathcal{D}$ is a λ system. The collection of subintervals of $[0,1]$ is a π-system which is contained in $\mathcal{D}$. Therefore $\mathcal{B}([0,1]) = \sigma(J : J \text{ is a subinterval }) \subseteq D$ by the $\pi - \lambda$ theorem. Since $\mathcal{D} \subseteq \mathcal{B}([0,1])$ by definition, we have $\mathcal{D} = \mathcal{B}([0,1])$.