Stochastic Stabilization of Dynamical Systems using Lévy Noise

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Abstract

We investigate the perturbation of the non-linear differential equation $\frac{dx(t)}{dt} = f(x(t))$ by random noise terms consisting of Brownian motion and an independent Poisson random measure. We find conditions under which the perturbed system is almost surely exponentially stable and estimate the corresponding Lyapunov exponents.

Keywords: stochastic differential equation, Lévy noise, Poisson random measure, Brownian motion, Lévy process, CGMY process, almost sure asymptotic stability, Lyapunov exponent, stabilization, destabilization

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1 Introduction

It is well-known that an unstable deterministic dynamical system can be stabilized when it is perturbed by noise. There have been a number of studies of this phenomenon using different types of noise and the following list is far from exhaustive: Bellman et al. [4], Meerkov [13], Khasminski [7] and Arnold et al. [3]. Mao in [11] and [12] pp. 135-141 has established a general theory of stochastic stabilization, using a multi-dimensional Brownian motion as the source of noise. To our knowledge there has been no systematic work so far in which the noise is a more general Lévy process. The purpose of this paper is to take some first steps in this direction, building extensively on Mao's results in the Brownian motion case. This seems to be timely as there has recently been extensive activity from the point of view of both theoretical development and applications of Lévy processes (see e.g. Applebaum [1] and references therein). The work presented here is a sequel to that in Applebaum and Siakalli [2] where we studied asymptotic stability for a class of stochastic differential equations (SDEs) with jumps.

We focus in this paper on a first order ordinary differential equation that is perturbed by Lévy noise, i.e. a Brownian motion and an independent Poisson random measure.

We couple the Brownian motion, small jump and large jumps terms separately to the system and investigate the almost sure asymptotic stability of the stochastic differential equation (SDE) that is so obtained.

The organisation of the paper is as follows. In section 2 we review some preliminary results on SDEs that we'll need. In section 3 we study stabilization for noise consisting of Brownian motion and small jumps. Large jumps are considered separately in section 4 and in section 5 we combine the results together. We are now able to consider perturbation by a generic Lévy process. We consider an example that is motivated by mathematical finance where the driving Lévy process is a Carr-Geman-Madan-Yor (or CGMY) process. In section 6 we examine the special case where an unstable linear system is perturbed by a Brownian motion and an independent Poisson process and we continue working with this noise in section 7 where we return to non-linear systems. Finally we present a new phenomenon that was suggested to us by Xuerong Mao. We know from Mao [12] section 4.5 that Brownian motion can destabilize a stable deterministic system and we have shown above that an unstable deterministic system can be stabilized by a Poisson process. We now show that an unstable system that is stabilized by Poisson noise can still be destabilized by Brownian motion when the dimension of the state space is at least two.

Notation. Throughout this paper $\mathbb{R}^+ := [0, \infty)$. The open ball of radius $r > 0$ in $\mathbb{R}^m$ that is centred on the origin is denoted by B_r and $\widehat{B_r} := B_r - \{0\}$. $\mathcal{M}_{d,m}(\mathbb{R})$ is the space of all real-valued $d \times m$ matrices and $\mathcal{M}_d(\mathbb{R}) := \mathcal{M}_{d,d}(\mathbb{R})$. We denote by $\|A\|$ the matrix norm defined by $\|A\| = \sup\{|Ay| : |y| = 1\}$ where $A \in \mathcal{M}_d(\mathbb{R})$. Note that the norm of a vector y in Euclidean space is always denoted $|y|$. We will frequently use the *logarithmic inequality*:

$$\log(x) \leq x - 1,$$

whenever $x > 0$.

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2 Preliminaries

2.1 Stochastic Differential Equations With Jumps

Suppose that we have the system of non-linear ordinary differential equations

$$\frac{dx(t)}{dt} = f(x(t)) \qquad \text{on} \quad t \geq t_0 \tag{1}$$

with $x(t_0) = x_0 \in \mathbb{R}^d$, where $f : \mathbb{R}^d \to \mathbb{R}^d$.

Assumption 2.1 We assume that $f : \mathbb{R}^d \to \mathbb{R}^d$ is a locally Lipschitz continuous function and furthermore, for some $K > 0$

$$|f(x)| \leq K|x| \quad \text{for all } x \in \mathbb{R}^d. \tag{2}$$

Let $(\Omega, \mathcal{F}, P)$ be a probability space that is equipped with a filtration $(\mathcal{F}_t, t \geq 0)$ which satisfies the usual hypotheses of right continuity and completeness. Suppose that we are given an m-dimensional standard $\mathcal{F}_t$-adapted Brownian motion process $B = (B(t), t \geq 0)$ with $B(t) = (B_1(t), \ldots, B_m(t))$ for each $t \geq 0$ and an independent $\mathcal{F}_t$-adapted Poisson random measure N defined on $\mathbb{R}^+ \times (\mathbb{R}^m \setminus \{0\})$ with compensator $\tilde{N}$ of the form $\tilde{N}(dt, dy) = N(dt, dy) - \nu(dy)dt$ where ν is a Lévy measure.

In the following, the stochastically perturbed system corresponding to (1) will have the form

$$\begin{aligned} dx(t) &= f(x(t-))dt + \sum_{k=1}^{m} G_k x(t-) dB_k(t) + \int_{|y|<r} D(y)x(t-)\tilde{N}(dt, dy) \\ &\quad + \int_{|y|\geq r} E(y)x(t-)N(dt, dy) \qquad (3) \end{aligned}$$

for all $t \geq t_0$, where $G_k \in \mathcal{M}_d(\mathbb{R})$, for $1 \leq k \leq m$, and D and E are suitable functions from $\mathbb{R}^m$ to $\mathcal{M}_d(\mathbb{R})$. The positive number r plays the role here of separating "small jumps" (which are compensated) from "large jumps" (which are not).

Given the initial condition $x(t_0) = x_0 \in \mathbb{R}^d$, the uniqueness and existence of the solution of the SDE (3) when $f : \mathbb{R}^d \to \mathbb{R}^d$ is a locally Lipschitz continuous function, is guaranteed from Theorem 2.2 that follows.

The unique solution to (3) is denoted by $x(t)$ for each $t \geq t_0$. Assume that $f(0) = 0$, then (3) has a solution $x(t) = 0$ for all $t \geq t_0$ corresponding to the initial value $x(t_0) = 0$, which is called the *trivial solution*.

Consider the following general d-dimensional SDE

$$dx(t) = f(x(t-))dt + g(x(t-))dB(t) + \int_{|y|<r} H(x(t-), y)\tilde{N}(dt, dy) \qquad (4)$$

on $t_0 \leq t \leq T$ with initial value $x(t_0) = x_0$, such that $x_0 \in \mathbb{R}^d$. Here $f : \mathbb{R}^d \to \mathbb{R}^d$, $g : \mathbb{R}^d \to \mathcal{M}_{d,m}(\mathbb{R})$ and $H : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$ are measurable mappings which will satisfy additional conditions to ensure existence and uniqueness hold (see below and also Chapter 6 of [1] for the general theory of such equations).

For the existence and uniqueness of solutions to (4) the following theorem applies.

Theorem 2.2 *Assume that local Lipschitz conditions and growth conditions for the coefficients of (4) hold. Under these conditions a unique càdlàg, adapted solution exists.*

Proof: This is by standard Picard iteration. Full details are given in Siakalli [15] pp. 78-83. Note that the local Lipschitz and growth conditions for H are precisely those given in Applebaum [1] p.376 and pp.366 (respectively.) $\square$

We will need the following technical result from Applebaum, Siakalli [2] to ensure that the solution of (4) can never reach the origin provided that $x_0 \neq 0$.

Assumption 2.3 We suppose that H is always such that

$$\nu\Big\{y \in \widehat{B_r}, \text{there exists } x \neq 0 \text{ such that } x + H(x,y) = 0\Big\} = 0.$$

We require that assumption 2.3 holds for the rest of this paper.

Lemma 2.4 Assume that for any $\theta > 0$ there exists $K_\theta > 0$, such that

$$|f(x)| + \|g(x)\|_1 + 2\int_{|y|<r} |H(x,y)| \left(\frac{|x| + |H(x,y)|}{|x + H(x,y)|}\right) \nu(dy) \leq K_\theta |x| \;\text{ if } |x| \leq \theta, \tag{5}$$

where $\|g(x)\|_1 := \sqrt{\sum_{i=1}^d \sum_{j=1}^m g_{ij}(x) g_{ji}(x)}$.

If $x_0 \neq 0$ then

$$P\left(x(t) \neq 0 \;\text{ for all } t \geq t_0\right) = 1. \tag{6}$$

Proof: See Applebaum, Siakalli [2].

We will need to generalize Lemma 2.4 to the case where the SDE includes all the noise components of a generic Lévy process. Let $K : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$ be a measurable mapping such that $z \to K(z,y)$ is continuous for each $y \in B_r^c$. Now let $z = (z(t), t \geq t_0)$ be the solution of the following SDE

$$dz(t) = f(z(t-))dt + g(z(t-))dB(t) + \int_{|y|<r} H(z(t-),y)\tilde{N}(dt,dy) + \int_{|y|\geq r} K(z(t-),y)N(dt,dy) \tag{7}$$

on $t \geq t_0$ with initial value $z(t_0) = z_0$, such that $z_0 \in \mathbb{R}^d$. Existence and uniqueness of càdlàg, adapted solutions to (7) follows from Applebaum [1] Theorem 6.2.9 p.374. In order to consider the extension of Lemma 2.4 to the case where "large jumps" are included, it is sufficient for the needs of this paper to consider the linear case $K(z,y) = E(y)z$, where E is a continuous map from $\mathbb{R}^m$ to $\mathcal{M}_d(\mathbb{R})$ and we assume this form from now on.

Lemma 2.5 Assume that the hypothesis of Lemma 2.4 holds and that $I + E(y)$ is invertible for all $y \in B_r(0)^c$. Then

$$P\left(z(t) \neq 0 \text{ for all } t \geq t_0\right) = 1.$$

Proof: This follows by a straightforward application of the interlacing technique as in Applebaum [1] Theorem 6.2.9 pp. 311, indeed if $(\tau_n, n \in \mathbb{N})$ are the jump times of the compound Poisson process $(P(t), t \geq t_0)$ where $P(t) := \int_{|y|\geq r} yN(t - t_0, dy)$ then for each $n \in \mathbb{N}$,

$$z(\tau_n) = (I + E(\Delta P(\tau_n))z(\tau_n-),$$

from which the result follows. $\square$

We remark that Li, Dong and Situ [9] pp. 123 have obtained the same result for more general jump-diffusion processes but using more complicated techniques.

2.2 Strong Law of Large Numbers for Local Martingales

We will frequently require the strong law of large numbers for local martingales as is found in Lipster [10] and we will briefly describe this for the reader's benefit. Let $(M(t), t \geq t_0)$ be a local martingale and for each $t \geq t_0$ define

$$\rho_M(t) := \int_{t_0}^{t} \frac{d\langle M\rangle(s)}{(1+s)^2},$$

where $\langle M\rangle(t) := \langle M, M\rangle(t)$ is *Meyer's angle bracket process.* The required strong law states that if $P\left(\lim_{t\to\infty} \rho_M(t) < \infty\right) = 1$ then $P\left(\lim_{t\to\infty} \frac{M(t)}{t} = 0\right) = 1$.

3 Stabilization Using Compensated Poisson Integrals

In this section we will mainly be concerned with the case where $H(x, y) = D(y)x$ in (4). So to satisfy assumption 2.3 we want $x + D(y)x = 0$ to have no non-trivial solutions (except on a set of possible ν-measure zero) which is true if and only if $D(y)$ does not have an eigenvalue of -1 (ν almost everywhere).

We will examine stability conditions for the non-linear deterministic system (1) which is perturbed by noise as follows:

$$dx(t) = f(x(t-))dt + \sum_{k=1}^{m} G_k x(t-)dB_k(t) + \int_{|y|<r} D(y)x(t-)\tilde{N}(dt, dy) \tag{8}$$

on $t \geq t_0$, $x(t_0) = x_0 \in \mathbb{R}^d$ where $G_k \in \mathcal{M}_d(\mathbb{R})$ for $1 \leq k \leq m$, and $y \to D(y)$ is a measurable map from $\mathbb{R}^m$ to $\mathcal{M}_d(\mathbb{R})$.

Assumption 3.1 We require that

$$\int_{|y|<r} \left(\|D(y)\| \vee \|D(y)\|^2\right) \nu(dy) < \infty,$$

and that $D(y)$ does not have any eigenvalue equal to -1 (ν almost everywhere).

The equation (8) has a unique solution by Theorem 2.2.

In the following theorem we will establish conditions on the coefficients of (8) for the trivial solution of the perturbed system (8) to be almost surely exponentially stable. In particular this demonstrates that the compensated Poisson noise can have a similar role to the Brownian motion (as in Mao [11] and [12] pp. 135-141) in stabilizing dynamical systems.

Theorem 3.2 *Let assumption 3.1 hold. Suppose that the following conditions are satisfied where* $\xi > 0, \gamma \geq 0,\ \delta \geq 0$

$$(i) \sum_{k=1}^{m} |G_k x|^2 \leq \xi |x|^2 \qquad \text{and} \qquad (ii) \sum_{k=1}^{m} |x^T G_k x|^2 \geq \gamma |x|^4$$

$$(iii) \int_{|y|<r} x^T D(y) x \, \nu(dy) \geq \delta |x|^2$$

for all $x \in \mathbb{R}^d$.

Then the sample Lyapunov exponent of the solution of (8) exists and satisfies

$$\limsup_{t\to\infty} \frac{1}{t} \log |x(t)| \leq -\left(\gamma - K - \frac{\xi}{2} - \int_{|y|<r} \log\left(1 + \|D(y)\|\right) \nu(dy) + \delta\right) \qquad a.s.$$

for any $x_0 \neq 0$. *If* $\gamma > K + \frac{\xi}{2} - \delta + \int_{|y|<r} \log\left(1 + \|D(y)\|\right) \nu(dy)$ *then the trivial solution of the system (8) is almost surely exponentially stable.*

Proof: Fix $x_0 \neq 0$. We first assume that (5) holds. Due to Lemma 2.4, then $x(t) \neq 0$ for all $t \geq t_0$ almost surely. Apply Itô's formula to $\log(|x(t)|^2)$. Then for each $t \geq t_0$

$$\begin{aligned}
\log(|x(t)|^2) &= \log(|x_0|^2) + \int_{t_0}^{t} \frac{2x(s-)^T}{|x(s-)|^2} f(x(s-))\, ds + \sum_{k=1}^{m} \int_{t_0}^{t} \frac{2x(s-)^T}{|x(s-)|^2} G_k x(s-)\, dB_k(s) \\
&\quad + \sum_{k=1}^{m} \int_{t_0}^{t} \left(\frac{|G_k x(s-)|^2}{|x(s-)|^2} - \frac{2|x(s-)^T G_k x(s-)|^2}{|x(s-)|^4} \right) ds \\
&\quad + \int_{t_0}^{t} \int_{|y|<r} \left[\log\left(|x(s-) + D(y)x(s-)|^2\right) - \log\left(|x(s-)|^2\right) \right] \tilde{N}(ds, dy) \\
&\quad + \int_{t_0}^{t} \int_{|y|<r} \left[\log\left(|x(s-) + D(y)x(s-)|^2\right) - \log\left(|x(s-)|^2\right) \right. \\
&\qquad \left. - \frac{2x(s-)^T D(y) x(s-)}{|x(s-)|^2} \right] \nu(dy) ds. \qquad (9)
\end{aligned}$$

The terms on the right hand side of the first two lines of this display are dealt with exactly as in Theorem 3.1 of Mao [11] and so we concentrate on the jumps terms. After some straightforward calculations and use of condition (iii) we obtain the following estimate for the last integral in (9):

$$\begin{aligned}
&\int_{t_0}^{t} \int_{|y|<r} \left[\log\left(|x(s-) + D(y)x(s-)|^2\right) - \log\left(|x(s-)|^2\right) - \frac{2x(s-)^T D(y) x(s-)}{|x(s-)|^2} \right] \nu(dy) ds \\
&\leq 2(t - t_0) \left[\int_{|y|<r} \log\left(1 + \|D(y)\|\right) \nu(dy) - \delta \right] \qquad (10)
\end{aligned}$$

where we note that $\int_{|y|<r} \log\left(1 + \|D(y)\|\right) \nu(dy) < \infty$, by the logarithmic inequality and assumption 3.1.

Now define $M(t) := \int_{t_0}^t \int_{|y|<r} \log\left(\frac{|x(s-) + D(y)x(s-)|^2}{|x(s-)|^2}\right) \tilde{N}(ds, dy)$, for each $t \geq t_0$. The process $(M(t), t \geq t_0)$ is a local martingale and following Kunita [8], p.323 we find that

$$\begin{aligned} \langle M \rangle(t) &= \int_{t_0}^t \int_{|y|<r} \left[\log\left(\frac{|x(s-) + D(y)x(s-)|^2}{|x(s-)|^2}\right)\right]^2 \nu(dy)ds \\ &\leq 4(t - t_0) \int_{|y|<r} ||D(y)||^2 \nu(dy), \end{aligned}$$

by elementary linear algebra and the logarithmic inequality. We then easily deduce that

$$\rho_M(t) \leq \frac{4(t - t_0)}{(1 + t_0)(1 + t)} \int_{|y|<r} ||D(y)||^2 \nu(dy),$$

and so by the law of large numbers for local martingales $\lim_{t\to\infty} \frac{M(t)}{t} = 0$ (a.s.). We thus obtain

$$\limsup_{t\to\infty} \frac{1}{t} \log|x(t)| \leq -\left(\gamma - K - \frac{\xi}{2} - \int_{|y|<r} \log\left(1 + \|D(y)\|\right) \nu(dy) + \delta\right) \qquad a.s.$$

If (5) does not hold, we may apply the stopping time argument that is used at the end of the proof of Theorem 3.7 in [2]. □

4 Stabilization by "Large Jump" Poisson Integrals

In this section we will perturb the non-linear system (1) by a Poisson random measure which represents the "large jumps" of a Lévy process i.e.

$$dx(t) = f(x(t-))dt + \int_{|y|\geq r} E(y)x(t-)N(dt, dy) \quad \text{on} \quad t \geq t_0 \tag{11}$$

with initial condition $x(t_0) = x_0 \in \mathbb{R}^d$ and where $y \to E(y)$ is a continuous map from $\mathbb{R}^m$ to $\mathcal{M}_d(\mathbb{R})$. The equation (11) has a unique global solution as described in section 2.

For the rest of this section we impose the following constraint:

Assumption 4.1 We require that

$$\int_{|y|\geq r} \|E(y)\|^2 \nu(dy) < \infty,$$

and that $E(y)$ does not have any eigenvalue equal to -1 (ν-almost everywhere).

Note that by assumption 4.1 the Cauchy-Schwarz inequality and the fact that $\nu(B_r^c) < \infty$, we have that $\int_{|y|\geq r} \|E(y)\|_1 \, \nu(dy) < \infty$. Hence we can write (11) as

$$dx(t) = \left(f(x(t-)) + \int_{|y|\geq r} E(y)x(t-)\nu(dy) \right) dt + \int_{|y|\geq r} E(y)x(t-)\tilde{N}(dt, dy).$$

We could now investigate stabilization by using Theorem 3.2, but we will instead develop the theory again as the structure of (11) enables new insights to be obtained.

Lemma 4.2

$$\sup_{x\in\mathbb{R}^d\setminus\{0\}} \int_{|y|\geq r} [\log(|x + E(y)x|) - \log(|x|)] \, \nu(dy) < \infty. \tag{12}$$

Proof: The result is easily deduced by using the logarithmic inequality and the fact that $\int_{|y|\geq r} \|E(y)\|\nu(dy) < \infty$. □

Next we will prove that the system (1) can be stabilized almost surely if is perturbed by the "large jumps" part of the Lévy noise.

Theorem 4.3 *Let assumption 4.1 hold. If there exists $K > 0$ such that*

$$\sup_{z\in\mathbb{R}^d\setminus\{0\}} \int_{|y|\geq r} \left[\log\left(\left|z + E(y)z\right|\right) - \log(|z|) \right] \nu(dy) < -K, \tag{13}$$

then the sample Lyapunov exponent of the solution of (11) exists and satisfies

$$\limsup_{t\to\infty} \frac{1}{t} \log|x(t)| \quad \leq \quad K + \sup_{z\in\mathbb{R}^d\setminus\{0\}} \int_{|y|\geq r} \left[\log(|z + E(y)z|) - \log(|z|) \right] \nu(dy).$$

Furthermore the trivial solution of the system (11) is almost surely exponentially stable.

Proof: Fix $x_0 \neq 0$. We assume that (5) holds. By Lemma 2.5 $x(t) \neq 0$ for all $t \geq t_0$ almost surely. Apply Itô's formula to $\log(|x(t)|^2)$. Then, by (12) for each $t \geq t_0$ we deduce that

$$\begin{aligned} \log(|x(t)|^2) &= \log(|x_0|^2) + \int_{t_0}^t \frac{2x(s-)^T}{|x(s-)|^2} f(x(s-)) \, ds \\ &\quad + \int_{t_0}^t \int_{|y|\geq r} \left[\log\left(|x(s-) + E(y)x(s-)|^2\right) - \log\left(|x(s-)|^2\right) \right] \tilde{N}(ds, dy) \\ &\quad + \int_{t_0}^t \int_{|y|\geq r} \left[\log\left(|x(s-) + E(y)x(s-)|^2\right) - \log\left(|x(s-)|^2\right) \right] \nu(dy)ds. \end{aligned}$$

By (2) then

$$\begin{aligned}\frac{1}{t}\log(|x(t)|^2) &\leq \frac{1}{t}\log(|x_0|^2) + 2K\frac{(t-t_0)}{t} + \frac{1}{t}\int_{t_0}^{t}\int_{|y|\geq r}\log\left(\frac{|x(s-)+E(y)x(s-)|^2}{|x(s-)|^2}\right)\tilde{N}(ds,dy)\\ &+ \frac{(t-t_0)}{t}\sup_{t_0\leq s\leq t}\int_{|y|\geq r}\left[\log\left(|x(s-)+E(y)x(s-)|^2\right) - \log\left(|x(s-)|^2\right)\right]\nu(dy).\end{aligned}$$

Letting $t \to \infty$ and applying the law of large numbers for local martingales we see that the sample Lyapunov exponent exists and satisfies

$$\limsup_{t\to\infty}\frac{1}{t}\log|x(t)| \leq K + \sup_{t_0\leq s\leq\infty}\int_{|y|\geq r}\left[\log\left(|x(s-)+E(y)x(s-)|\right) - \log\left(|x(s-)|\right)\right]\nu(dy).$$

To obtain that the trivial solution of the system (11) is almost surely exponentially stable $\limsup_{t\to\infty}\frac{1}{t}\log|x(t)| < 0$ must hold. By (13) the required result follows. If (5) fails we use a stopping time argument to obtain the result. □

Condition (13) is the most general that we can find but it is not immediately clear how to find mappings that satisfy it. In the following we present a very simple example but first we need a useful lemma.

Lemma 4.4 If $z \in \mathbb{R}^d$ with $z \neq 0$ and F is a $d \times d$ real valued symmetric matrix then

$$-\infty < \log\left(|z + Fz|\right) < \log\left(|z|\right) \tag{14}$$

if and only if the eigenvalues of F belong to the set $(-2,-1) \cup (-1,0)$.

Proof: By monotonicity of the logarithmic function, the right hand side of (14) holds if and only if $|z|^2 > |z + Fz|^2$, i.e. $2\langle z, Fz\rangle + |Fz|^2 < 0$. Equivalently $\langle z, (F^2 + 2F)z\rangle < 0$ and the result follows easily from here. □

By Lemma 4.4 we can ensure that the left-hand side of the inequality (13) is negative if $E(y)$ is symmetric and all the eigenvalues of $E(y)$ belong to the set $(-2,-1) \cup (-1,0)$.

Example 4.5 Fix $r = d = m = 1$ and let $E(y) = b$ for all $y \in \mathbb{R}$, where $b \in (-2,-1) \cup (-1,0)$. Now for any Lévy measure ν we have that

$$\int_{|y|\geq 1}\log(|1+b|)\nu(dy) = \log(|1+b|)\nu(|y|\geq 1) = R\log(|1+b|) < 0$$

where $R = \nu(|y| \geq 1) < \infty$.

We require that (13) is satisfied. Hence, if $0 < K < -R\log(|1+b|)$ then the trivial solution of

$$\begin{aligned} dx(t) &= f(x(t-))dt + \int_{|y|\geq 1} bx(t-)N(dt,dy) \\ &= f(x(t-))dt + bx(t-)dN(t) \end{aligned}$$

is almost surely exponentially stable since (13) clearly holds. In the last display $(N(t), t \geq t_0)$ is the Poisson process of intensity $\nu(B_1^c)$ defined by $N(t) := N(t, B_1^c)$.

5 Stabilization of SDEs driven by Lévy Noise

In this section we are interested in finding stability conditions for SDEs driven by generic Lévy noise i.e.

$$\begin{aligned} dx(t) &= f(x(t-))dt + \sum_{k=1}^{m} G_k x(t-)dB_k(t) + \int_{|y|<r} D(y)x(t-)\tilde{N}(dt,dy) \\ &\quad + \int_{|y|\geq r} E(y)x(t-)N(dt,dy) \end{aligned} \tag{15}$$

on $t \geq t_0$ and initial condition $x(t_0) = x_0 \in \mathbb{R}^d$.

Combining the results of Theorem 3.2 and Theorem 4.3 we obtain the following.

Theorem 5.1 *Assume that the following conditions are satisfied where $\xi > 0$, $\gamma \geq 0$, $\delta \geq 0$*

$$(i) \sum_{k=1}^{m} |G_k x|^2 \leq \xi |x|^2 \qquad \text{and} \qquad (ii) \sum_{k=1}^{m} |x^T G_k x|^2 \geq \gamma |x|^4$$

$$(iii) \int_{|y|<r} x^T D(y) x \, \nu(dy) \geq \delta |x|^2$$

for all $x \in \mathbb{R}^d$,

$$\int_{|y|<r} \left(\|D(y)\| \vee \|D(y)\|^2 \right) \nu(dy) < \infty \qquad \text{and} \qquad \int_{|y|\geq r} \|E(y)\|^2 \nu(dy) < \infty$$

where neither of D or $E : \mathbb{R}^m \to \mathcal{M}_d(\mathbb{R})$ have an eigenvalue equal to -1 (ν-almost everywhere).
Then the sample Lyapunov exponent of the solution of (15) exists and satisfies

$$\limsup_{t\to\infty} \frac{1}{t} \log |x(t)| \quad \leq \quad -\left(\gamma - K - \frac{\xi}{2} - \int_{|y|<r} \log\left(1 + \|D(y)\|\right) \nu(dy) + \delta - M(r) \right)$$

where $M(r) := \sup_{z \in \mathbb{R}^d \setminus \{0\}} \int_{|y|\geq r} \left[\log\left(|z + E(y)z|\right) - \log\left(|z|\right) \right] \nu(dy) < \infty$.

If $\gamma > K + \frac{\xi}{2} - \delta + \int_{|y|<r} \log(1 + \|D(y)\|)\nu(dy) + M(r)$ then the trivial solution of (15) is almost surely exponentially stable.

Proof: Immediate from the proofs of Theorem 3.2 and Theorem 4.3.

5.1 SDEs driven by Lévy Processes

Consider the following stochastically perturbed system

$$dx(t) = f(x(t-))dt + \sum_{i=1}^{m} G_i x(t-) dY^i(t) \tag{16}$$

where $Y = (Y(t), t \geq t_0)$ is a Lévy process taking values in $\mathbb{R}^m$. Let $G_i \in \mathcal{M}_d(\mathbb{R})$ for $1 \leq i \leq m$.

We denote the Lévy-Itô decomposition of $Y = (Y(t), t \geq t_0)$ as

$$Y^i(t) = b^i t + B_A^i(t) + \int_{|y|<1} y^i \tilde{N}(t, dy) + \int_{|y|\geq 1} y^i N(t, dy)$$

for each $1 \leq i \leq m$, $t \geq t_0$ where $b \in \mathbb{R}^m$, $(B_A(t), t \geq t_0)$ is an m-dimensional Brownian motion with covariance matrix A and for each $1 \leq i \leq m$, $t \geq t_0$ $B_A^i(t) = \sum_{k=1}^{p} \sigma^{ik} B_k(t)$ where $B_1, \ldots, B_p$ are standard one-dimensional Brownian motions and σ is an $m \times p$ real-valued matrix for which $\sigma\sigma^T = A$ (see e.g. Applebaum [1] pp. 112).

Assumption 5.2

$$(i) \int_{\mathbb{R}^m \setminus \{0\}} |y|^2 \nu(dy) < \infty \qquad \text{and} \qquad (ii) \int_{|y|<1} |y| \nu(dy) < \infty.$$

We require that assumption 5.2 holds for the rest of this subsection.

Remark 5.3 Note that (i) in assumption 5.2 is a necessary and sufficient condition for a Lévy process to have a finite second moment (see e.g. Applebaum [1] Theorem 2.5.2 p.132) and (ii) is a necessary and sufficient condition to have finite variation if $A = 0$ holds (see e.g. Applebaum [1] Theorem 2.4.25 p.129).

By assumption 5.2 we can write the Lévy-Itô decomposition of $Y = (Y(t), t \geq t_0)$ for each $1 \leq i \leq m$, $t \geq t_0$ as

$$Y^i(t) = \tilde{b}^i t + B_A^i(t) + \int_{\mathbb{R}^m \setminus \{0\}} y^i \tilde{N}(t, dy)$$

where $\tilde{b}^i = b^i + \int_{|y|\geq 1} y^i \nu(dy)$. Hence (16) can be written as

$$\begin{aligned} dx(t) \quad = \quad & \left(f(x(t-)) + \sum_{i=1}^{m} G_i x(t-) \tilde{b}^i \right) dt + \sum_{k=1}^{p} G'_k x(t-) dB_k(t) \\ & + \sum_{i=1}^{m} \int_{\mathbb{R}^m \setminus \{0\}} G_i x(t-) y^i \tilde{N}(dt, dy). \end{aligned} \tag{17}$$

where $G'_k = \sum_{i=1}^m G_i \sigma^{ik}$.

By assumption 5.2 it follows that the requirements on the coefficient of the compensated Poisson integral (assumption 3.1) are fulfilled. Also assumption 2.1 is satisfied. Hence, fitting the coefficients of (17) to the conditions of Theorem 3.2 we have the following.

Corollary 5.4 Let an SDE be of the form (16) with G_i, for $1 \le i \le m$ fixed and suppose that assumption 2.3 holds. If the following conditions are satisfied where $\xi > 0, \gamma \ge 0, \delta \ge 0$

$$(i) \sum_{k=1}^p |G'_k x|^2 \le \xi |x|^2 \qquad \text{and} \qquad (ii) \sum_{k=1}^p |x^T G'_k x|^2 \ge \gamma |x|^4$$

$$(iii) \textstyle\sum_{i=1}^m \int_{\mathbb{R}^m \setminus \{0\}} y^i x^T G_i x \, \nu(dy) \ge \delta |x|^2$$

for all $x \in \mathbb{R}^d$ then the sample Lyapunov exponent of the solution of (16) exists and satisfies

$$\limsup_{t\to\infty} \frac{1}{t} \log |x(t)| \quad \le \quad -\left(\gamma - K - \frac{\xi}{2} - \sum_{i=1}^m \int_{\mathbb{R}^m \setminus \{0\}} \log\left(1 + \|G_i\||y^i|\right) \nu(dy) + \delta\right) \qquad a.s.$$

for any $x_0 \neq 0$. If $\gamma > K + \frac{\xi}{2} + \sum_{i=1}^m \int_{\mathbb{R}^m \setminus \{0\}} \log\left(1 + \|G_i\||y^i|\right) \nu(dy) - \delta$ then the trivial solution of the system (16) is almost surely exponentially stable. □

In the following we will demonstrate the theory that we have developed for SDEs of the form (16), by giving an example that is motivated by modelling stock price movements in financial mathematics (see Cont and Tankov [5] and Carr et al. [6]).

Example 5.5 (CGMY process)
Let $m = 1$ and choose the driving Lévy process to be CGMY process. The CGMY process is a pure jump process with Lévy measure

$$\nu(dy) = \begin{cases} C' \frac{\exp(-Q|y|)}{|y|^{1+\alpha}} & \text{for } y < 0 \\ \\ C \frac{\exp(-M|y|)}{|y|^{1+\alpha}} & \text{for } y > 0 \end{cases} \tag{18}$$

where $C > 0$, $C' > 0$, $Q > 0$, $M > 0$ and $0 \le \alpha < 2$. For $0 \le \alpha < 1$ the process has finite variation and for $1 < \alpha < 2$ has infinite variation (for further details see Carr et al. [5]).

For simplicity we assume that $C' = 0$ (so the CGMY process has only positive jumps) and $0 \le \alpha < 1$. Assume that $d = 1$ and $G > 0$. In this case, by Lemma 2.5, the solution is almost surely always non-zero. Since the CGMY process is a pure jump process there is no Brownian motion component and therefore we will only need to check if assumption

5.2, and condition (iii) of Corollary 5.4 are satisfied. Assumption 5.2 (i) holds since

$$\int_{\mathbb{R}-\{0\}} |y|^2 \nu(dy) = C \int_{0+}^{\infty} y^2 \frac{e^{-My}}{y^{1+\alpha}} dy = CM^{\alpha-2}\Gamma(2-\alpha) < \infty.$$

In order to find a value for δ and check assumption 5.2 (ii) we compute

$$\int_{\mathbb{R}\setminus\{0\}} |y|\nu(dy) \quad = \quad \int_0^{\infty} y^{-\alpha} C e^{-My} dy = CM^{\alpha-1}\Gamma(1-\alpha) < \infty, \tag{19}$$

and we see that condition (iii) is satisfied with $\delta = GCM^{\alpha-1}\Gamma(1-\alpha) > 0$.

It follows that the sample Lyapunov exponent of the solution of (16) exists and satisfies

$$\limsup_{t\to\infty} \frac{1}{t} \log|x(t)| \quad \leq \quad K + \int_0^{\infty} \log\left(1 + \|G\|y\right) \nu(dy) - GCM^{\alpha-1}\Gamma(1-\alpha) \qquad a.s.$$

If $K + \int_0^{\infty} \log\left(1 + Gy\right) \nu(dy) - GCM^{\alpha-1}\Gamma(1-\alpha) < 0$ then the trivial solution of (16) is almost surely exponentially stable.

6 Stochastic stabilization of linear systems

In this section we will consider a special case of (1), a linear unstable one-dimensional deterministic system. We will investigate if it is possible to stabilize this system by adding a mixture of Brownian motion and Poisson noise. Of course this case is already covered by Theorem 3.2 but we will find that using a single Poisson process allows us to have greater flexibility. For simplicity we take $t_0 = 0$.
Consider the following unstable one-dimensional system

$$\frac{dx(t)}{dt} = ax(t) \qquad \text{on } t \geq 0$$

with initial condition $x(0) = x_0 \in \mathbb{R}$ and $a > 0$. Suppose that we perturb the system with noise and the system now has the following form:

$$dx(t) = ax(t-)dt + bx(t-)dB(t) + cx(t-)d\tilde{N}(t) \tag{20}$$

where $b > 0$, $c > -1$. Here $(B(t), t \geq 0)$ is a one-dimensional Brownian motion and $(\tilde{N}(t), t \geq 0)$ is the compensated Poisson process with $\tilde{N}(t) = N(t) - \lambda t$ where $\lambda > 0$ is the intensity of the Poisson process $(N(t), t \geq 0)$. Assume that the processes $(B(t), t \geq 0)$ and $(N(t), t \geq 0)$ are independent.

Hence,

$$dx(t) \quad = \quad x(t-)dZ(t) \tag{21}$$

where $Z(t) = (a - \lambda c)t + bB(t) + cN(t)$, for each $t \geq 0$.

It follows that that $x(t) = x_0\mathcal{E}_Z(t)$, where $\mathcal{E}_Z$ is the stochastic exponential of Z (see Applebaum [1] pp. 249), with $x(0) = x_0 \in \mathbb{R}$. For simplicity we take $x_0 = 1$ from now on.

Using simple properties of stochastic exponentials see e.g. Applebaum [1] p.283) and the laws of large numbers for a Brownian motion and for a Poisson process it follows that

$$\limsup_{t\to\infty} \frac{1}{t} \log |x(t)| = \left(a - \lambda c - \frac{1}{2}b^2\right) + \lambda \log(1+c). \qquad a.s. \tag{22}$$

If $\left(a - \lambda c - \frac{1}{2}b^2\right) + \lambda \log(1+c) < 0$, then the perturbed system (21) becomes almost surely stable. For further details see Siakalli [15] pp. 99-101.
From (22) it is easy to see that a one dimensional system that is stabilised by a Poisson process can never be destabilised by Brownian motion. In section 7.2 below, we will see that different behaviour is possible in higher dimensions.

7 Perturbation of non-linear Systems by Brownian Motion and Poisson Processes

In this section we explore further insights that are obtained when the Poisson random measure reduces to a (compensated) Poisson process of intensity λ.

Let the non-linear system (1) be perturbed by Brownian motion and an independent compensated Poisson process as is shown below.

$$dx(t) = f(x(t-))dt + \sum_{k=1}^{m} G_k x(t-)dB_k(t) + Dx(t-)d\tilde{N}(t) \qquad \text{on} \qquad t \geq t_0$$

i.e.

$$dx(t) = [f(x(t-)) - \lambda Dx(t-)]dt + \sum_{k=1}^{m} G_k x(t-)dB_k(t) + Dx(t-)dN(t) \tag{23}$$

where $D \in \mathcal{M}_d(\mathbb{R})$.

7.1 Stabilization

Let $x = (x(t), t \geq t_0)$ be the solution of

$$dx(t) = h(x(t-))dt + \sum_{k=1}^{m} G_k(x(t-))dB_k(t) + Dx(t-)dN(t) \qquad \text{on } \ t \geq t_0 \tag{24}$$

where $h(x) = f(x) - \lambda Dx$. From now on we assume that D is symmetric and that -1 is not an eigenvalue of D. It follows that $I + D$ is invertible.

The result that follows is a special case of Theorem 3.2. However we present a separate proof below. The fact that D does not depend on the jumps allows us to get more direct and simple results.

Theorem 7.1 *Suppose that the following conditions are satisfied for all $x \in \mathbb{R}^d$, where $\xi > 0, \gamma \geq 0$*

$$(i) \sum_{k=1}^{m} |G_k x|^2 \leq \xi |x|^2 \qquad \text{and} \qquad (ii) \sum_{k=1}^{m} |x^T G_k x|^2 \geq \gamma |x|^4 \tag{25}$$

and D is a $d \times d$ symmetric positive definite matrix. Then the sample Lyapunov exponent of the solution of (23) exists and satisfies

$$\limsup_{t \to \infty} \frac{1}{t} \log |x(t)| \leq -\left(\gamma + \lambda \mu_{min} - K - \frac{\xi}{2} - \lambda \log\left(1 + \mu_{max}\right)\right) \qquad a.s.$$

for any $x_0 \neq 0$, where μ_{min} and μ_{max} are the minimum and maximum eigenvalues of the matrix D (respectively). In particular the trivial solution of (23) is almost surely exponentially stable if the following relationship is satisfied

$$\gamma + \lambda \mu_{min} - K - \tfrac{\xi}{2} - \lambda \log\left(1 + \mu_{max}\right) > 0.$$

Proof: Fix any $x_0 \neq 0$ and assume (5). Apply Itô's formula to $\log(|x(t)|^2)$. Since D is symmetric we have the following estimate

$$\begin{aligned}
\log(|x(t)|^2) \;\leq\; & \log(|x_0|^2) + \int_{t_0}^{t} \frac{2x(s-)^T}{|x(s-)|^2} f(x(s-))\, ds - 2\lambda \mu_{min}(t - t_0) \\
& + \sum_{k=1}^{m} \int_{t_0}^{t} \frac{2x(s-)^T}{|x(s-)|^2} G_k x(s-) dB_k(s) \\
& + \sum_{k=1}^{m} \int_{t_0}^{t} \left(\frac{|G_k x(s-)|^2}{|x(s-)|^2} - \frac{2|x(s-)^T G_k x(s-)|^2}{|x(s-)|^4} \right) ds \\
& + \log\left(1 + \mu_{max}\right)^2 \left(N(t) - N(t_0)\right).
\end{aligned}$$

Applying conditions (2) and (25), we obtain

$$\begin{aligned}
\frac{1}{t} \log(|x(t)|^2) \;\leq\; & \frac{1}{t} \log(|x_0|^2) + \frac{(t - t_0)}{t} (2K - 2\lambda \mu_{min} + \xi - 2\gamma) \\
& + \frac{1}{t} \sum_{k=1}^{m} \int_{t_0}^{t} \frac{2x(s-)^T}{|x(s-)|^2} G_k x(s-)\, dB_k(s) \\
& + 2 \log\left(1 + \mu_{max}\right) \frac{N(t) - N(t_0)}{t}.
\end{aligned}$$

Using the strong law of large numbers for the Brownian motion integral and the well known law of large numbers for Poisson processes (see Sato [14] pp. 246) it follows that

$$\limsup_{t \to \infty} \frac{1}{t} \log(|x(t)|^2) \leq (2K - 2\lambda \mu_{min} + \xi - 2\gamma) + 2\lambda \log\left(1 + \mu_{max}\right) \qquad a.s.,$$

and the required result follows. If (5) fails to hold, the result is still valid by a stopping time argument. □

7.2 Stochastic destabilization of non-linear systems.

Assume that (1) is unstable. We have seen in section 7.1 that we can stabilize (1) by adding Poisson noise. To be precise consider the following stochastically perturbed system

$$dx(t) = f(x(t-))dt + Dx(t-)d\tilde{N}(t) \qquad \text{on} \qquad t \geq t_0 \tag{26}$$

which is a special case of (23). Suppose that D is a $d \times d$ symmetric positive definite matrix and

$$\lambda\mu_{min} - K - \lambda \log\big(1 + \mu_{max}\big) > 0, \tag{27}$$

holds, where μ_{min} and μ_{max} are the minimum and maximum eigenvalues of the matrix D respectively and λ is the intensity of the Poisson process. Then by Theorem 7.1, the trivial solution of (26) is almost surely exponentially stable.

In the following we will investigate the conditions under which (26) is destabilized when it is perturbed by Brownian motion. So we consider the following SDE

$$dx(t) = f(x(t-))dt + Dx(t-)d\tilde{N}(t) + \sum_{k=1}^{m} G_k x(t-)dB_k(t) \qquad \text{on} \qquad t \geq t_0. \tag{28}$$

Theorem 7.2 *Assume that D is a $d \times d$ symmetric positive definite matrix with μ_{min} and μ_{max} its minimum and maximum eigenvalues respectively and (27) holds. If the following conditions are satisfied for all $x \in \mathbb{R}^d$ where $\xi > 0, \gamma \geq 0$*

$$(i) \sum_{k=1}^{m} |G_k x|^2 \geq \xi |x|^2 \qquad\qquad (ii) \sum_{k=1}^{m} |x^T G_k x|^2 \leq \gamma |x|^4 \tag{29}$$

then

$$\liminf_{t\to\infty} \frac{1}{t} \log|x(t)| \geq \left(-K - \lambda\mu_{max} + \frac{\xi}{2} - \gamma + \lambda \log\big(1 + \mu_{min}\big)\right) \qquad a.s.$$

for any $x_0 \neq 0$. In particular if $\xi > 2K + 2\lambda\mu_{max} + 2\gamma - 2\lambda\log\big(1 + \mu_{min}\big)$ then the trivial solution of (28) tends to infinity almost surely exponentially fast.

Proof: Fix $x_0 \neq 0$. Due to Lemma 2.5 then $x(t) \neq 0$ for all $t \geq t_0$. Applying Itô's formula to $\log(|x(t)|^2)$, conditions (2) and (29) and using the fact that D is symmetric we have

$$\begin{aligned}\log(|x(t)|^2) \;\;\geq\;\; & \log(|x_0|^2) + 2\sum_{k=1}^{m} \int_{t_0}^{t} \frac{x(s-)^T G_k x(s-)}{|x(s-)|^2}\, dB_k(s) \\ & + (-2K - 2\lambda\mu_{max} + \xi - 2\gamma)(t - t_0) + \log\Big((1 + \mu_{min})^2\Big)(N(t) - N(t_0)).\end{aligned}$$

Using the same arguments as in the proof of Theorem 7.1 we deduce that

$$\liminf_{t\to\infty} \frac{1}{t} \log |x(t)|^2 \geq \Big(-2K - 2\lambda\mu_{max} + \xi - 2\gamma + 2\lambda \log\big(1+\mu_{min}\big)\Big)$$

and the required result follows. □

The following example is considered by Mao (see [11] pp. 287 and [12] pp. 140-141) for the case that (1) is perturbed by Brownian motion. We are now able to include random jumps in the model.

Example 7.3 Consider the following stochastic perturbation of the system (26).

$$dx(t) = f(x(t-))dt + \sum_{i=1}^{p} D_i x(t-)d\tilde{N}_i(t) + \sum_{k=1}^{m} G_k x(t-)dB_k(t) \tag{30}$$

with initial condition $x(t_0) = x_0 \in \mathbb{R}^d$ and $d = 2p$ $(p \geq 1)$. Here $(B(t), t \geq t_0)$ is an m-dimensional Brownian motion where for each $t \geq t_0$ $B(t) = (B_1(t), B_2(t), \ldots, B_m(t))$ with $B_1, B_2, \ldots, B_m$ independent one-dimensional Brownian motions and $(N(t), t \geq t_0)$ is a p-dimensional Poisson process with $N_1, \ldots, N_p$ being one-dimensional independent Poisson processes which are all independent of the Brownian motion. Let G_k for $1 \leq k \leq m$ be a $d \times d$ constant matrix with $G_k = 0$ for $2 \leq k \leq m$ and

$$G_1 = \begin{pmatrix} 0 & \sigma & \ldots & 0 \\ -\sigma & 0 & \ldots & \\ \ldots & & & \\ \ldots & 0 & 0 & \sigma \\ 0 & \ldots & -\sigma & 0 \end{pmatrix}$$

where σ is a constant and D_1 be a $d \times d$ diagonal matrix such that $D_1 = \mathrm{diag}(a, \cdots, a)$, where $a > 0$ and $D_i = 0$ for $2 \leq i \leq p$. As a result the system has the form of

$$dx(t) = f(x(t-))dt + D_1 x(t-)d\tilde{N}_1(t) + G_1 x(t-)dB_1(t)$$

and it becomes

$$dx(t) = f(x(t-)) + ax(t-)d\tilde{N}_1(t) + \sigma y(t-)dB_1(t) \tag{31}$$

where $y(t)^T = (x_2(t), -x_1(t), \ldots, x_{2p}(t), -x_{2p-1}(t))$ for each $t \geq t_0$.

The hypotheses (i) and (ii) of Theorem 7.2 are satisfied since

$$(i) \sum_{k=1}^{m} |G_k x|^2 = \sigma^2 |x|^2 \qquad \text{and} \qquad (ii) \sum_{k=1}^{m} |x^T G_k x|^2 = 0.$$

as in Mao [11] pp. 287 and [12] pp. 140-141. Hence,

$$\liminf_{t\to\infty} \frac{1}{t} \log |x(t)| \geq \Big(\frac{1}{2}\sigma^2 - K - \lambda a + \lambda \log(1+a) \Big) \qquad a.s. \tag{32}$$

If $\sigma^2 > 2K + 2\lambda a - 2\lambda \log(1+a)$ then the trivial solution of (31) is almost surely exponentially unstable.

Remark 7.4 From the above calculations, we see that any stable system of the form $dy(t) = f(y(t-))dt + Dy(t-)d\tilde{N}(t)$ can be destabilized by Brownian motion provided that the dimension of the system is $d \geq 2$ and (2) and (27) are satisfied.

References

[1] D.Applebaum, *Lévy Processes and Stochastic Calculus* second edition, Cambridge University Press (2009)

[2] D.Applebaum, M.Siakalli *Asymptotic stability properties of stochastic differential equations driven by Lévy noise*, *Journal of Applied Probability* **46**, 1116-29 (2009)

[3] L. Arnold, H. Crauel, V. Wihstutz, Stabilization of linear systems by noise, *SIAM Journal on Control and Optimization*, **21**, 451-461, (1983)

[4] R. Bellman, J. Bentsman, S. Meerkov, Stability for fast periodic systems, *IEEE Transactions on Automatic Control*, **AC-30**, 289-291, (1985)

[5] P. Carr, H. Geman, D. B. Madan, M. Yor, The fine structure of asset returns: an empirical investigation, *Journal of Business* **75**, 305–33 (2002)

[6] R.Cont, P.Tankov, *Financial Modelling with Jump Processes*, Chapman and Hall/CRC (2004)

[7] R. Z. Khasminski, *Stochastic Stability of Differential Equations*, Alphen: Sojtjoff and Noordhoff (1980), translation of the Russian edition (1969)

[8] H.Kunita, Stochastic differential equations based on Lévy processes and stochastic flows of diffeomorphisms, in *Real and Stochastic Analysis, New Perspectives*, M.M.Rao (ed.), 305-75 Birkhaüser Boston Basel Berlin (2004)

[9] C. W. Li, Z. Dong, R. Situ, Almost sure stability of linear stochastic differential equations with jumps, *Probability Theory and Related Fields*, **123**, 121-155, (2002)

[10] R. Lipster, A strong law of large numbers for local martingales, *Stochastics*, **3**, 217-228 (1980)

[11] X. Mao, Stochastic stabilization and destabilization, *Systems and Control Letters*, **23**, 279-90 (1994)

[12] X.Mao, *Stochastic Differential Equations and Applications*, Horwood Publications, Chichester (1997), second edition (2008)

[13] S. Meerkov, Condition of vibrational stabilizability for a class of non-linear systems *IEEE Transactions on Automatic Control*, **27**, 485-487, (1982)

[14] K. Sato, *Lévy process and Infinitely Divisible Distributions*, Cambridge University Press (1999)

[15] M.Siakalli, Stability properties of stochastic differential equations driven by Lévy noise, University of Sheffield PhD thesis (2009)