Lectures on Lévy Processes, Stochastic Calculus and Financial Applications, Ovronnaz September 2005

3 Lecture 3: From Girsanov's theorem to option pricing with Lévy processes and Malliavin calculus

3.1 Lévy-type Stochastic Integrals as Martingales

We continue to work with real-valued Lévy-type stochastic integrals of the form

$$dY(t) = G(t)dt + F(t)dB(t) + H(t,x)\tilde{N}(dt,dx) + K(t,x)N(dt,dx). \quad (3.1)$$

Our first goal is to find necessary and sufficient conditions for a Lévy-type stochastic integral Y to be a martingale. At least formally, we would expect this to be the case if the first and fourth term vanish with probability one, since the second and third terms are martingales.

First we impose some conditions on K and G. For simplicity we make these stronger than necessary in order to avoid having to work with the general notion of *local martingale*.

(M1) $\mathbb{E}\left(\int_0^t \int_{|x|\geq 1} |K(s,x)|^2 \nu(dx)ds\right) < \infty$.

(M2) $\int_0^t \mathbb{E}(|G(s)|)ds < \infty$ for each $t > 0$.

(Note that $E = \hat{B} - \{0\} = \{x \in \mathbb{R}^d; 0 < |x| < 1\}$ throughout this section.)

It follows from (M1) and the Cauchy-Schwarz inequality, that $\int_0^t \int_{|x|\geq 1} |K(s,x)|\nu(dx)ds < \infty$ (a.s.) and we can write

$$\int_0^t \int_{|x|\geq 1} K(s,x)N(dx,ds) = \int_0^t \int_{|x|\geq 1} K(s,x)\tilde{N}(dx,ds) + \int_0^t \int_{|x|\geq 1} K(s,x)\nu(dx)ds,$$

for each $t \geq 0$, with the compensated integral being a martingale.

Theorem 3.1 *If Y is a Lévy-type stochastic integral of the form (3.1) and the assumptions (M1) and (M2) are satisfied, then Y is a martingale if and only if*

$$G(t) + \int_{|x|\geq 1} K(t,x)\nu(dx) = 0 \quad (a.s.),$$

for (Lebesgue) almost all $t \geq 0$.

Proof. For each $0 \leq s < t < \infty$,

$$\begin{aligned} Y(t) &= Y(s) + \int_s^t F(u)dB(u) + \int_s^t \int_{|x|<1} H(u,x)\tilde{N}(du,dx) \\ &+ \int_s^t \int_{|x|>1} K(u,x)\tilde{N}(du,dx) + \int_s^t \left(G(u) + \int_{|x|\geq 1} K(u,x)\nu(dx) \right) du. \end{aligned}$$

Now assume that $(Y(t), t \geq 0)$ is a martingale, then we must have

$$\mathbb{E}_s \left[\int_s^t \left(G(u) + \int_{|x|\geq 1} K(u,x)\nu(dx) \right) du \right] = 0.$$

Using the fact that by (M1) and (M2),

$$\begin{aligned} \left| \int_s^t \left(G(u) + \int_{|x|\geq 1} K(u,x)\nu(dx) \right) du \right| &\leq \int_0^t \left| G(u) + \int_{|x|\geq 1} K(u,x)\nu(dx) \right| du \\ &< \infty \text{ (a.s.)}, \end{aligned}$$

and the conditional version of dominated convergence, we deduce that

$$\mathbb{E}_s \int_s^t \left(G(u) + \int_{|x|\geq 1} K(u,x)\nu(dx) \right) du = 0.$$

(M1) and (M2) ensure that we can use the conditional Fubini theorem to obtain

$$\int_s^t \mathbb{E}_s \left(G(u) + \int_{|x|\geq 1} K(u,x)\nu(dx) \right) du = 0.$$

It follows that $\displaystyle \lim_{h\to 0} \frac{1}{h} \int_s^{s+h} \mathbb{E}_s \left(G(u) + \int_{|x|\geq 1} K(u,x)\nu(dx) \right) du = 0,$

and hence by Lebesgue's differentiation theorem we have

$$\mathbb{E}_s \left(G(s) + \int_{|x|\geq 1} K(s,x)\nu(dx) \right) = 0,$$

for (Lebesgue) almost all $s \geq 0$. But $G(\cdot) + \int_{|x|\geq 1} K(\cdot,x)\nu(dx)$ is adapted and the result follows. The converse is immediate. □

3.2 Exponential Martingales

We continue to study Lévy-type stochastic integrals satisfying (M1) to (M2). We now turn our attention to the process $e^Y = (e^{Y(t)}, t \geq 0)$.

By Itô's formula, we find for each $t \geq 0$,

$$\begin{aligned} e^{Y(t)} &= 1 + \int_0^t e^{Y(s-)} F(s) dB(s) \\ &+ \int_0^t \int_{|x|<1} e^{Y(s-)} (e^{H(s,x)} - 1) \tilde{N}(ds, dx) + \int_0^t \int_{|x| \geq 1} e^{Y(s-)} (e^{K(s,x)} - 1) \tilde{N}(ds, dx) \\ &+ \int_0^t e^{Y(s-)} \left(G(s) + \frac{1}{2} F(s)^2 + \int_{|x|<1} (e^{H(s,x)} - 1 - H(s,x)) \nu(dx) \right. \\ &+ \left. \int_{|x| \geq 1} (e^{K(s,x)} - 1) \nu(dx) \right) ds. \end{aligned} \tag{3.2}$$

Hence, by an immediate application of Theorem 3.1, we have:

Corollary 3.1 *e^Y is a martingale if and only if*

$$G(s) + \frac{1}{2} F(s)^2 + \int_{|x|<1} (e^{H(s,x)} - 1 - H(s,x)) \nu(dx) + \int_{|x| \geq 1} (e^{K(s,x)} - 1) \nu(dx) = 0, \tag{3.3}$$

almost surely and for (Lebesgue) almost all $s \geq 0$.

So e^Y is a martingale if and only if for (Lebesgue almost all) $t \geq 0$,

$$\begin{aligned} e^{Y(t)} &= 1 + \int_0^t e^{Y(s-)} F(s) dB(s) + \int_0^t \int_{|x|<1} e^{Y(s-)} (e^{H(s,x)} - 1) \tilde{N}(ds, dx) \\ &+ \int_0^t \int_{|x| \geq 1} e^{Y(s-)} (e^{K(s,x)} - 1) \tilde{N}(ds, dx). \end{aligned} \tag{3.4}$$

Furthermore, since each stochastic integral in (3.4) has zero mean, we deduce that for all $t \geq 0$,

$$\mathbb{E}(e^{Y(t)}) = 1.$$

Since $e^{Y(t)} > 0$ for all $t \geq 0$, we can use it to define a new probability measure Q_t which is absolutely continuous with respect to P, and has Radon-Nikodym derivative $\frac{dQ_t}{dP} = e^{Y(t)}$. We'll make explicit use of this below.

The process e^Y given by equation (3.4) is called an *exponential martingale*. Two important examples are:

1. **The Brownian Case**

   Here Y is a Brownian integral of the form

   $$Y(t) = \int_0^t F(s)dB(s) + \int_0^t G(s)ds,$$

   for each $t \geq 0$. The unique solution to (3.3) is $G(t) = -\frac{1}{2}F(t)^2$ (a.e.). We then have, for each $t \geq 0$

   $$e^{Y(t)} = \exp\left(\int_0^t F(s)dB(s) - \frac{1}{2}\int_0^t F(s)^2 ds\right).$$

2. **The Poisson Case**

   Here Y is a Poisson integral driven by a Poisson process N of intensity λ and has the form

   $$Y(t) = \int_0^t K(s)dN(s) + \int_0^t G(s)ds,$$

   for each $t \geq 0$.

   The unique solution to (3.3) is $G(t) = -\lambda\int_0^t (e^{K(s)} - 1)ds$ (a.e.). For each $t \geq 0$, we obtain

   $$e^{Y(t)} = \exp\left(\int_0^t K(s)dN(s) - \lambda\int_0^t (e^{K(s)} - 1)ds\right).$$

3.3 Change of Measure - Girsanov's Theorem

If we are given two distinct probability measures P and Q on $(\Omega, \mathcal{F})$, we will write $\mathbb{E}_P$ ($\mathbb{E}_Q$) to denote expectation with respect to P (respectively Q). We also use the terminology P-martingale, P-Brownian motion etc, when we want to emphasise that P is the operative measure. We remark that Q and P are each also probability measures on $(\Omega, \mathcal{F}_t)$, for each $t \geq 0$, and we will use the notations Q_t and P_t when the measures are restricted in this way. Suppose that $Q << P$, then each $Q_t << P_t$ and we sometimes write $\left.\frac{dQ}{dP}\right|_t = \frac{dQ_t}{dP_t}$.

In the proof of the next lemma, we will make extensive use of the defining property of the conditional expectation $\mathbb{E}(X|\mathcal{G})$ of an integrable $\mathcal{F}$-measurable random variable X on to a sub-σ-algebra $\mathcal{G}$, i.e. if Y is an integrable $\mathcal{G}$-measurable random variable for which

$$\mathbb{E}(\chi_A Y) = \mathbb{E}(\chi_A X) \text{ for all } A \in \mathcal{G},$$
$$\text{then} \quad Y = \mathbb{E}(X|\mathcal{G}) \quad \text{a.s.}$$

Lemma 3.1 $\left(\left.\frac{dQ}{dP}\right|_t, t \geq 0 \right)$ *is a P-martingale.*

Proof. For each $t \geq 0$, let $M(t) = \left.\frac{dQ}{dP}\right|_t$. For all $0 \leq s \leq t, A \in \mathcal{F}_s$,

$$\begin{aligned} \mathbb{E}_P(\chi_A \mathbb{E}_P(M(t)|\mathcal{F}_s)) &= \mathbb{E}_P(\chi_A M(t)) \\ &= \mathbb{E}_{P_t}(\chi_A M(t)) = \mathbb{E}_{Q_t}(\chi_A) \\ &= \mathbb{E}_{Q_s}(\chi_A) = \mathbb{E}_{P_s}(\chi_A M(s)) \\ &= \mathbb{E}_P(\chi_A M(s)). \end{aligned}$$

□

Now let e^Y be an exponential martingale. Then since

$$\mathbb{E}_P(e^{Y(t)}) = \mathbb{E}_{P_t}(e^{Y(t)}) = 1$$

we can define a probability measure Q_t on $(\Omega, \mathcal{F}_t)$ by

$$\frac{dQ_t}{dP_t} = e^{Y(t)}, \tag{3.5}$$

for each $t \geq 0$.

From now on, we will find it convenient to fix a time interval $[0, T]$. We write $P = P_T$ and $Q = Q_T$.

Before we establish Girsanov's theorem, which is the key result of this section, we need a useful lemma.

Lemma 3.2 *$M = (M(t), 0 \leq t \leq T)$ is a Q-martingale if and only if $Me^Y = (M(t)e^{Y(t)}, 0 \leq t \leq T)$ is a P-martingale.*

Proof. Let $A \in \mathcal{F}_s$ and assume that M is a Q-martingale, then for each $0 \leq s < t < \infty$,

$$\begin{aligned} \int_A M(t)e^{Y(t)}dP &= \int_A M(t)e^{Y(t)}dP_t = \int_A M(t)dQ_t \\ &= \int_A M(t)dQ = \int_A M(s)dQ = \int_A M(s)dQ_s \\ &= \int_A M(s)e^{Y(s)}dP_s = \int_A M(s)e^{Y(s)}dP. \end{aligned}$$

The converse is proved in the same way. □

In the following we take Y to be a Brownian integral, so that for each $0 \leq t \leq T$,

$$e^{Y(t)} = \exp\left(\int_0^t F(s)dB(s) - \frac{1}{2}\int_0^t F(s)^2 ds\right)$$

We define a new process $W = (W(t), 0 \leq t \leq T)$ by

$$W(t) = B(t) - \int_0^t F(s)ds,$$

for each $t \geq 0$.

Theorem 3.2 (Girsanov) *W is a Q-Brownian motion.*

Proof. First we use Itô's product formula to find that for each $0 \leq t \leq T$,

$$\begin{aligned} d(W(t)e^{Y(t)}) &= dW(t)e^{Y(t)} + W(t)de^{Y(t)} + dW(t)de^{Y(t)} \\ &= e^{Y(t)}dB(t) - e^{Y(t)}F(t)dt + W(t)e^{Y(t)}F(t)dB(t) + e^{Y(t)}F(t)dt \\ &= e^{Y(t)}[1 + W(t)F(t)]dB(t). \end{aligned}$$

Hence We^Y is a P-martingale and so (by Lemma 3.2), W is a Q-martingale. Moreover since $W(0) = 0$ (a.s.), we see that W is centered (with respect to Q).

But $dW(t)^2 = dt$ and so $[W, W](t) = t$. The result now follows from Lévy's characterisation of Brownian motion. □

The following partial extension of Girsanov's theorem will be of use to us in applications to finance:-

Let $M = (M(t), 0 \leq t \leq T)$ be a martingale of the form

$$M(t) = \int_0^t \int_{|x|<1} L(x, s)\tilde{N}(ds, dx),$$

where $L \in \mathcal{P}(t, E)$. Let e^Y be an exponential martingale. By Lemma 3.2 it follows that $N = (N(t), 0 \leq t \leq T)$ is a martingale where

$$N(t) = M(t) - \int_0^t \int_{|x|<1} L(x, s)(e^{H(s,x)} - 1)\nu(dx)ds.$$

To see this, argue as in the first part of the proof of theorem 3.2 and apply Itô's product formula to the process Ne^Y.

3.4 Stochastic Calculus and Lévy Processes in Option Pricing

In recent years, there has been a growing interest in the use of Lévy processes and discontinuous semimartingales to model market behaviour and not only are these of great mathematical interest but there is growing evidence that they may be more realistic models than those that insist on continuous sample paths. Our aim in this section is to give a brief introduction to some of these ideas.

Stock prices (we deal with a single stock, for simplicity) will be modelled as a random process $(S(t), t \geq 0)$. All options which we will consider will be modelled as *contingent claims* with maturity date $T > 0$. This means that Z is a non-negative $\mathcal{F}_T$ measurable random variable which represents the value of the option to its holder. In the following k is the *exercise price* which is the fixed price which the option entitles the holder to pay for one unit of stock, irrespective of its market value. Some examples:

- *European call option*

$$Z = \max\{S(T) - k, 0\} := (S(T) - k)^+$$

.

- *American call option*

$$Z = \sup_{0 \leq \tau \leq T} \max\{S(\tau) - k, 0\}$$

.

- *Asian option*

$$Z = \max\left\{\frac{1}{T}\int_0^T (S(t) - k)dt, 0\right\}.$$

We assume that the interest rate r is constant so that a principal P invested in a bank account at time 0 grows to Pe^{-rt} by time t. So a sum of money Q promised at time t has a *discounted value* Qe^{-rt} today. In particular, if $(S(t), t \geq 0)$ is the stock price, we define the *discounted process* $\tilde{S} = (\tilde{S}(t), t \geq 0)$ where each $\tilde{S}(t) = e^{-rt}S(t)$.

3.4.1 Portfolios, Arbitrage, Martingales

An investor (which may be an individual or a company) will hold their investments as a combination of risky stocks and cash in the bank, say. Let $\alpha(t)$ and $\beta(t)$ denote the amount each of these which we hold, respectively, at time t. The pair of adapted processes (α, β) where $\alpha = (\alpha(t), t \geq 0)$ and $\beta = (\beta(t), t \geq 0)$ is called a *portfolio* or *trading strategy* . The total value of all our investments at time t is denoted as $V(t)$, so

$$V(t) = \alpha(t)S(t) + \beta(t)ß(t).$$

One of the key aims of the Black-Scholes approach to option pricing is to be able to *hedge* the risk involved in selling options, by being able to construct a portfolio whose value at the expiration time T is exactly that of the option. To be precise, a portfolio is said to be *replicating* if

$$V(T) = Z.$$

Clearly, replicating portfolios are desirable objects. Another class of interesting portfolios are those that are *self-financing*, i.e. any change in wealth V is due only to changes in the values of stocks and bank accounts and not to any injections of capital from outside. We can model this using stochastic differentials if we make the assumption that the stock price process S is a semimartingale. We can then write

$$\begin{aligned} dV(t) &= \alpha(t)dS(t) + \beta(t)dß(t) \\ &= \alpha(t)dS(t) + r\beta(t)ß(t)dt. \end{aligned}$$

so the infinitesimal change in V arises solely through infinitesimal changes in S and ß. Notice how we have sneakily slipped Itô calculus into the picture by the assumption that $dS(t)$ should be interpreted in the Itô sense. This is absolutely crucial. If we try to use any other integral (e.g. Lebesgue-Stieltjes), then the theory which follows will certainly no longer work.

A market is said to be *complete* if every contingent claim can be replicated by a self-financing portfolio. So in a complete market, every option can be hedged by a portfolio which requires no outside injections of capital between its starting time and the expiration time.

An *arbitrage opportunity* exists if the market allows risk-free profit. Conversely the market is *arbitrage free* if there exists no self-financing strategy (α, β) for which $V(0) = 0, V(T) \geq 0$ and $P(V(T) > 0) > 0$.

At least in discrete time, we have the following remarkable result which illustrates how the absence of arbitrage forces the mathematical modeller into the world of stochastic analysis.

Theorem 3.3 (Fundamental Theorem of Asset Pricing 1) *If the market is free of arbitrage opportunities, then there exists a probability measure Q, which is equivalent to P, with respect to which the discounted process $\tilde{S}$ is a martingale.*

A similar result holds in the continuous case but we need to make more technical assumptions - instead of absence of arbitrage we need the stronger NFLVR hypothesis ("no free lunch with vanishing risk"). The philosophy of Theorem 3.3 will play a central role in the sequel.

Again in discrete time we have:

Theorem 3.4 (Fundamental Theorem of Asset Pricing 2) *An arbitrage-free market is complete if and only if there exists a unique probability measure Q, which is equivalent to P, with respect to which the discounted process $\tilde{S}$ is a martingale.*

Theorems 3.3 and 3.4 identify a key mathematical problem for us is to find a (unique, if possible) Q, which is equivalent to P, under which $\tilde{S}$ is a martingale. Such a Q is called a *martingale measure* or *risk-neutral measure*. If Q exists, but is not unique, then the market is said to be *incomplete*.

A martingale $M = (M(t), 0 \leq t \leq T)$ has the *predictable representation property* if for any $\mathcal{F}_T$ measurable X with $\mathbb{E}(|X|^2) < \infty$, there exists a predictable $(G(s), 0 \leq s \leq T)$ such that

$$X = \mathbb{E}(X) + \int_0^T G(s) dM(s).$$

If the discounted stock price is a Q-martingale which has the predictable representation property then the market is complete, i.e. we can always find a replicating self-financing strategy. Unfortunately the only Lévy processes which have this property are Brownian motion and the compensated Poisson process, consequently the generic Lévy market (see below) is incomplete.

Note. Let $B = (B(t), t \geq 0)$ be a one-dimensional Brownian motion and

$$g(t) = \sup\{0 \leq s \leq t; B(s) = 0\}.$$

Azéma's martingale

$$M(t) = \sqrt{\frac{\pi}{2}} \text{sign}(B(t)) \sqrt{t - g(t)}$$

is a martingale with respect to the filtration $\mathcal{F}_t = \sigma\{M(s); 0 \leq s \leq t\}$. It has the predictable representation property but it is not a Lévy process. (It is however a "quantum Lévy process").

In a complete market, it turns out that we have

$$V(t) = e^{-r(T-t)}\mathbb{E}_Q(X|\mathcal{F}_t),$$

and this is the arbitrage-free price of the claim X at time t.

3.5 Stock Prices as Lévy Process

So far we have said little about the key process S which models the evolution of stock prices. As far back as 1900, Bachelier in his PhD thesis proposed that this should be a Brownian motion . Indeed, this can be intuitively justified on the basis of the central limit theorem if one perceives the movement of stocks as due to the "invisible hand of the market" manifested as a very large number of independent, identically distributed decisions. One immediate problem with this is that it is unrealistic, as stock prices cannot become negative but Brownian motion can. An obvious way out of this is to take exponentials, but let us be more specific.

Financial analysts like to study the *return* on their investment, which in a small time interval $[t, t + \delta t]$ will be

$$\frac{\delta S(t)}{S(t)} = \frac{S(t + \delta t) - S(t)}{S(t)},$$

and it is then natural to directly introduce the noise at this level and write

$$\frac{\delta S(t)}{S(t)} = \sigma \delta X(t) + \mu \delta t,$$

where $X = (X(t), t \geq 0)$ is a semimartingale and σ, μ are parameters called the *volatility* and *stock drift* respectively. $\sigma > 0$ controls the strength of the coupling to the noise while $\mu \in \mathbb{R}$ represents deterministic effects, indeed if $\mathbb{E}(\delta X(t)) = 0$ for all $t \geq 0$ then μ is the mean rate of return.

We now interpret this in terms of Itô calculus by formally replacing all small changes which are written in terms of δ by Itô differentials. We then find that

$$\begin{aligned} dS(t) &= \sigma S(t-)dX(t) + \mu S(t-)dt \\ &= S(t-)dZ(t), \end{aligned} \tag{3.6}$$

where $Z(t) = \sigma X(t) + \mu t$.

We see immediately that $S(t) = \mathcal{E}_Z(t)$ is the stochastic exponential of the semimartingale Z as described in section 4.1. Indeed, when X is a standard Brownian motion $B = (B(t), t \geq 0)$, we obtain *geometric Brownian motion*, which has been very widely used as a model for stock prices since it was first proposed in 1965 by the economist Paul Samuelson

$$S(t) = \exp\left(\sigma B(t) + \left(\mu t - \frac{1}{2}\sigma^2 t\right)\right). \tag{3.7}$$

There has recently been a great deal of interest in taking X to be a Lévy process. One argument in favour of this is that stock prices clearly don't move continuously and a more realistic approach is one that allows small jumps in small time intervals. Moreover, empirical studies of stock prices indicate distributions with heavy tails, which are incompatible with a Gaussian model.

We will make the assumption from now on that X is indeed a Lévy process. Note immediately that in order for stock prices to be non-negative, (SE) yields $\Delta X(t) > -\sigma^{-1}$ (a.s.) for each $t > 0$ and for convenience, we will denote $c = -\sigma^{-1}$ in the sequel. We will also impose the following condition on the Lévy measure ν, $\int_{(c,-1]\cup[1,\infty)} x^2\nu(dx) < \infty$. This means that each $X(t)$ has finite first and second moments which seems to be a reasonable assumption for stock returns.

By the Lévy-Itô decomposition, for each $t \geq 0$, we have

$$X(t) = mt + \kappa B(t) + \int_c^\infty x\tilde{N}(t, dx), \tag{3.8}$$

where $\kappa \geq 0$, and in terms of the earlier parametrisation, $m = b + \int_{(c,-1]\cup[1,\infty)} x\nu(dx)$. So as to keep the notation simple, it is assumed in (3.8), and in the discussion that follows, that 0 is omitted from the range of integration.

Modelling $S(t)$ as $\mathcal{E}_Z(t)$, we obtain the following representation for the stock prices from (2.22):

$$\begin{aligned} d(\log(S(t))) &= \kappa\sigma dB(t) + (m\sigma + \mu - \frac{1}{2}\kappa^2\sigma^2)dt \qquad (3.9)\\ &+ \int_c^\infty \log(1+\sigma x)\tilde{N}(dt, dx) + \int_c^\infty (\log(1+\sigma x) - \sigma x)\nu(dx)dt. \end{aligned}$$

Some advantages of modelling with Lévy processes:

- Incomplete markets are believed to be more realistic.

- The Lévy-Itô decomposition gives a natural description of stock movements as jumps from the very small (infinite activity) to the very large (finite activity).
- There are a number of explicit mathematically tractable and realistic models e.g. variance-gamma, normal inverse Gaussian, hyperbolic etc.
- Some empirical studies indicate the presence of "heavy tails". Many Lévy processes have this property. Brownian motion has "light tails".

3.6 Change of Measure

Motivated by the philosophy behind the fundamental theorem of asset pricing (Theorems 3.3 and 3.4), we seek to find measures Q, which are equivalent to P, with respect to which the discounted stock process $\tilde{S}$ is a martingale. Rather than consider all possible changes of measure, we work in a restricted context where we can exploit our understanding of stochastic calculus based on Lévy processes.

Let Y be a Lévy-type stochastic integral which takes the form

$$dY(t) = G(t)dt + F(t)dB(t) + \int_{\mathbb{R}-\{0\}} H(t,x)\tilde{N}(dt,dx),$$

where in particular $H \in \mathcal{P}_2(t, \mathbb{R} - \{0\})$ for each $t \geq 0$. Note that we have deliberately chosen a restricted form of Y to be compatible with that of the Lévy process X in order to simplify the discussion below.

We consider the associated exponential process e^Y and we assume that this is a martingale (and so G is determined by F and H). Hence we can define a new measure Q by the prescription $\frac{dQ}{dP} = e^{Y(T)}$. Furthermore by Girsanov's theorem and the extension to jump integrals, for each $0 \leq t \leq T, A \in \mathcal{B}([c,\infty))$,

$$B_Q(t) = B(t) - \textstyle\int_0^t F(s)ds \text{ is a } Q\text{- Brownian motion.}$$

$$\tilde{N}_Q(t,A) = \tilde{N}(t,A) - \nu_Q(t,A) \text{ is a } Q\text{-martingale, where}$$
$$\nu_Q(t,A) = \textstyle\int_0^t \int_A (e^{H(s,x)} - 1)\nu(dx)ds.$$

Note that $\mathbb{E}_Q(\tilde{N}_Q(t,A)^2) = \int_0^t \int_A \mathbb{E}_Q(e^{H(s,x)})\nu(dx)ds$

We rewrite the discounted stock price $\tilde{S}(t) = e^{-rt}S(t)$ in terms of these new processes to find

$$d(\log(\tilde{S}(t))) \;\; = \;\; \kappa\sigma dB_Q(t) + \biggl(m\sigma + \mu - r - \frac{1}{2}\kappa^2\sigma^2 + \kappa\sigma F(t)$$

$$
\begin{aligned}
&+ \quad \sigma \int_{\mathbb{R}-\{0\}} x(e^{H(t,x)} - 1)\nu(dx)\Big) dt \\
&+ \quad \int_c^\infty \log(1+\sigma x)\tilde{N}_Q(dt,dx) \\
&+ \quad \int_c^\infty (\log(1+\sigma x) - \sigma x)\nu_Q(dt,dx). \qquad (3.10)
\end{aligned}
$$

Now write $\tilde{S}(t) = \tilde{S}_1(t)\tilde{S}_2(t)$, where

$$
\begin{aligned}
d(\log(\tilde{S}_1(t))) &= \kappa\sigma dB_Q(t) - \frac{1}{2}\kappa^2\sigma^2 dt \\
&+ \int_c^\infty \log(1+\sigma x)\tilde{N}_Q(dt,dx) \\
&+ \int_c^\infty (\log(1+\sigma x) - \sigma x)\nu_Q(dt,dx),
\end{aligned}
$$

and

$$
d(\log(\tilde{S}_2(t))) = \left(m\sigma + \mu - r + \kappa\sigma F(t) + \sigma \int_{\mathbb{R}-\{0\}} x(e^{H(t,x)} - 1)\nu(dx)\right) dt.
$$

On applying Itô's formula to $\tilde{S}_1$, we obtain

$$
d\tilde{S}_1(t) = \kappa\sigma\tilde{S}_1(t-)dB_Q(t) + \int_{(c,\infty)} \sigma\tilde{S}_1(t-)x\tilde{N}_Q(dt,dx).
$$

So $\tilde{S}_1$ is a Q-martingale, and hence $\tilde{S}$ is a Q-martingale if and only if

$$
m\sigma + \mu - r + \kappa\sigma F(t) + \sigma \int_{\mathbb{R}-\{0\}} x(e^{H(t,x)} - 1)\nu(dx) = 0 \quad \text{a.s.} \qquad (3.11)
$$

Now equation (3.11) clearly has an infinite number of possible solution pairs (F, H). To see this suppose that $f \in L^1(\mathbb{R} - \{0\}, \nu)$, then if (F, H) is a solution, so too is $\left(F + \int_{\mathbb{R}-\{0\}} f(x)\nu(dx), \log\left(e^H - \frac{\kappa f}{x}\right)\right)$. Consequently there are an infinite number of possible measures Q with respect to which $\tilde{S}$ is a martingale. So the general Lévy process model gives rise to incomplete markets. The following example is of considerable interest.

The Brownian Case

Here we have $\nu \equiv 0, \kappa \neq 0$ and the unique solution to (3.11) is

$$
F(t) = \frac{r - \mu - m\sigma}{\kappa\sigma} \quad \text{a.s.}.
$$

So in this case, the stock price is a geometric Brownian motion, and in fact we have a complete market. This leads to the famous *Black-Scholes pricing formula* which dates back to the pioneering work of Black and Scholes in 1973.

The Poisson case

Here we take $\kappa = 0$ and $\nu = \lambda\delta_1$ for $\lambda > m + \frac{\mu - r}{\sigma}$. Writing $H(t,1) = H(t)$, we find

$$H(t) = \log\left(\frac{r - \mu + (\lambda - m)\sigma}{\lambda\sigma}\right) \quad \text{(a.s.)}.$$

3.7 Incomplete Markets

If the market is complete and if there is a suitable predictable representation theorem available, it is clear that the Black-Scholes approach described above can, in principle, be applied to price contingent claims. However, if stock prices are driven by a general Lévy process as in (3.8), the market will be incomplete. Provided there are no arbitrage opportunities, we know that equivalent measures Q exist with respect to which $\tilde{S}$ will be a martingale, but these will no longer be unique. In this section we briefly examine some approaches which have been developed for incomplete markets. These involve finding a "selection principle" to reduce the class of all possible measures Q to a subclass, within which a unique measure can be found.

1. *The Föllmer-Schweizer Minimal Measure*

   In the Black-Scholes set-up, we have a unique martingale measure Q for which $\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = e^{Y(t)}$ where $d(e^{Y(t)}) = e^{Y(t)}F(t)dB(t)$ for $0 \leq t \leq T$. In the incomplete case, one approach to selecting Q would be to simply replace B by the martingale part of our Lévy process (3.8), so that we have

$$d(e^{Y(t)}) = e^{Y(t)}P(t)\left(\kappa dB(t) + \int_{(c,\infty)} x\tilde{N}(ds,dx)\right), \qquad (3.12)$$

   for some adapted process $P = (P(t), t \geq 0)$. If we compare this with the usual coefficients of exponential martingales in (3.2), we see that we have

$$\kappa P(t) = F(t), \qquad xP(t) = e^{H(t,x)} - 1$$

for each $t \geq 0, x > c$. Substituting these conditions into (3.11) yields

$$P(t) = \frac{r + \mu - m\sigma}{\sigma(\kappa^2 + \rho)}$$

where $\rho = \int_c^\infty x^2 \nu(dx)$, so this procedure selects a unique martingale measure, under the constraint that we only consider measure changes of the type (3.12). Terence Chan has shown that this coincides with a general procedure introduced by Föllmer and Schweitzer, which works by constructing a replicating portfolio of value $V(t) = \alpha(t)S(t) + \beta(t)ß(t)$ and discounting it to obtain $\tilde{V}(t) = \alpha(t)\tilde{S}(t) + \beta(t)ß(0)$. If we now define the cumulative cost $C(t) = \tilde{V}(t) - \int_0^t \alpha(s)d\tilde{S}(s)$, then Q minimises the risk $\mathbb{E}((C(T) - C(t))^2|\mathcal{F}_t)$.

2. *The Esscher Transform*

In this section, we will make the additional assumption that

$$\int_{|x|\geq 1} e^{ux}\nu(dx) < \infty,$$

for all $u \in \mathbb{R}$. In this case we can analytically continue the Lévy-Khintchine formula to obtain, for each $t \geq 0$

$$\mathbb{E}(e^{-uX(t)}) = e^{-t\psi(u)},$$

$$\text{where} \quad \psi(u) = -\eta(iu) = bu - \frac{1}{2}\kappa^2 u^2 + \int_c^\infty (1 - e^{-uy} - uy\chi_B(y))\nu(dy).$$

Now recall the martingales $M_u = (M_u(t), t \geq 0)$, where each $M_u(t) = \exp\{iuX(t) - t\eta(u)\}$. You can check directly that the martingale property is preserved under analytic continuation, and we will write $N_u(t) = M_{iu}(t) = \exp\{-uX(t) + t\psi(u)\}$. The key distinction between the martingales M_u and N_u is that the former are complex-valued while the latter are strictly positive. For each $u \in \mathbb{R}$, we may thus define a new probability measure by the prescription

$$\left.\frac{dQ_u}{dP}\right|_{\mathcal{F}_t} = N_u(t),$$

for each $0 \leq t \leq T$. Q_u is called the *Esscher transform* of P by N_u. It has a long history of application within actuarial science. Applying Itô's formula to N_u, we obtain

$$dN_u(t) = N_u(t-)(-\kappa u B(t) + (e^{-ux} - 1)\tilde{N}(dt, dx)). \tag{3.13}$$

On comparing this with our usual prescription (3.2) for exponential martingales e^Y, we find that

$$F(t) = -\kappa u, \qquad H(t, x) = -ux$$

and so (3.11) yields the following condition for Q_u to be a martingale measure:-

$$-\kappa^2 u\sigma + m\sigma + \mu - r + \sigma \int_c^\infty x(e^{-ux} - 1)\nu(dx) = 0.$$

Define $z(u) = \int_c^\infty x(e^{-ux} - 1)\nu(dx) - \kappa^2 u$, for each $u \in \mathbb{R}$. Then our condition takes the form

$$z(u) = \frac{r - \mu - m\sigma}{\sigma}.$$

Since $z'(u) \leq 0$, we see that z is monotonic decreasing and hence invertible. Hence, this choice of u yields a martingale measure, under the constraint that we only consider changes of measure of the form (3.13).

Chan has shown that this Q_u minimises the relative entropy $H(Q|P)$ where

$$H(Q|P) = \int \frac{dQ}{dP} \log\left(\frac{dQ}{dP}\right) dP.$$

3.8 Hyperbolic Lévy Processes in Finance

So far we have concentrated our efforts in general discussions about Lévy processes as models of stock prices, without looking at any particular case other than Brownian motion. An example of a Lévy process which appears to be well-suited to modelling stock price movements is the hyperbolic process which we will now describe.

3.8.1 Hyperbolic Distributions

Let $\Upsilon \in \mathcal{B}(\mathbb{R})$ and let $(g_\theta; \theta \in \Upsilon)$ be a family of probability density functions on $\mathbb{R}$ such that the mapping $(x, \theta) \to g_\theta(x)$ is jointly measurable from $\mathbb{R} \times \Upsilon$ to $\mathbb{R}$. Let ρ be another probability distribution on Υ which we call the *mixing measure*, then by Fubini's theorem, we see that the *probability mixture*

$$h(x) = \int_\Upsilon g_\theta(x)\rho(d\theta),$$

yields another probability density function h on $\mathbb{R}$. Indeed each $h(x) \geq 0$ and

$$\int_{\mathbb{R}} h(x)dx = \int_\Upsilon \left(\int_{\mathbb{R}} g_\theta(x)dx \right) \rho(d\theta) = \int_\Upsilon \rho(d\theta) = 1.$$

The hyperbolic distributions which we will now introduce arise exactly in this manner. First we need to describe the mixing measure ρ.

We begin with the following integral representation for Bessel functions of the third kind:-

$$K_\nu(x) = \frac{1}{2}\int_0^\infty u^{\nu-1} \exp\left(-\frac{1}{2}x\left(u + \frac{1}{u}\right)\right)du,$$

where $x, \nu \in \mathbb{R}$.

From this, we see immediately that $f_\nu^{a,b}$ is a probability density function on $(0, \infty)$, for each $a, b > 0$, where

$$f_\nu^{a,b}(x) = \frac{\left(\frac{a}{b}\right)^{\frac{\nu}{2}}}{2K_\nu(\sqrt{ab})} x^{\nu-1} \exp\left(-\frac{1}{2}\left(ax + \frac{b}{x}\right)\right).$$

The distribution which this represents is called a *generalised inverse Gaussian* and denoted $GIG(\nu, a, b)$. It clearly generalises the inverse Gaussian distribution discussed in our first lecture . In our probability mixture, we now take ρ to be a $GIG(1, a, b), \Upsilon = (0, \infty)$ and g_{σ^2} to be the probability density function of a $N(\mu + \beta\sigma^2, \sigma^2)$ where $\mu, \beta \in \mathbb{R}$. A straightforward but tedious computation, where we apply the beautiful result $K_{\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}}e^{-x}$, yields

$$h_{\delta,\mu}^{\alpha,\beta}(x) = \frac{\sqrt{\alpha^2 - \beta^2}}{2\alpha\delta K_1(\delta\sqrt{\alpha^2 - \beta^2})} \exp\{-\alpha\sqrt{\delta^2 + (x-\mu)^2} + \beta(x-\mu)\}, \quad (3.14)$$

for all $x \in \mathbb{R}$, where we have, in accordance with the usual convention, introduced the parameters $\alpha^2 = a + \beta^2$ and $\delta^2 = b$.

The corresponding law is called a *hyperbolic distribution* as $\log(h_{\delta,\mu}^{\alpha,\beta})$ is a hyperbola. These distributions were first introduced by O.Barndorff-Nielsen within models for the distribution of particle size in wind-blown sand deposits. They are infinitely divisible and, in fact, self-decomposable.

All the moments of a hyperbolic distribution exist and we may compute the moment generating function $M_{\delta,\mu}^{\alpha,\beta}(u) = \int_{\mathbb{R}} e^{ux} h_{\delta,\mu}^{\alpha,\beta}(x)dx$, to obtain:

Proposition 3.1 *For* $|u+\beta| < \alpha$,

$$M_{\delta,\mu}^{\alpha,\beta}(u) = e^{\mu u} \frac{\sqrt{\alpha^2 - \beta^2}}{K_1(\delta\sqrt{\alpha^2 - \beta^2})} \frac{K_1(\delta\sqrt{\alpha^2 - (\beta+u)^2})}{\sqrt{\alpha^2 - (\beta+u)^2}}.$$

Proof. Straightforward manipulation. □

Note that by analytic continuation, we get the characteristic function $\phi(u) = M(iu)$ which is valid for all $u \in \mathbb{R}$.

For simplicity, we will restrict ourselves to the symmetric case where $\mu = \beta = 0$. If we reparametrise, using $\zeta = \delta\alpha$, we obtain the two-parameter family of densities

$$h_{\zeta,\delta}(x) = \frac{1}{2\delta K_1(\zeta)} \exp\left\{-\zeta\sqrt{1 + \left(\frac{x}{\delta}\right)^2}\right\}.$$

The corresponding Lévy process $X_{\zeta,\delta} = (X_{\zeta,\delta}(t), t \geq 0)$ has no Gaussian part and can be written

$$X_{\zeta,\delta}(t) = \int_0^t \int_{\mathbb{R}-\{0\}} x\tilde{N}(ds, dx),$$

for each $t \geq 0$.

3.8.2 Option Pricing with Hyperbolic Lévy Processes

The hyperbolic Lévy process was first applied to option pricing by Eberlein and Keller following a suggestion by O.Barndorff-Nielsen. The analogy with sand production was that just as large rocks are broken down to smaller and smaller particles to create sand so, to quote Bingham and Kiesel "this 'energy cascade effect' might be paralleled in the 'information cascade effect', whereby price-sensitive information originates in, say, a global newsflash and trickles down through national and local level to smaller and smaller units of the economic and social environment."

We may again model the stock price $S = (S(t), t \geq 0)$ as a stochastic exponential driven by a $X_{\zeta,\delta}$ so that

$$dS(t) = S(t-)dX_{\zeta,\delta}(t),$$

for each $t \geq 0$ (we omit volatility for now and return to this point later). A drawback of this approach is that the jumps of $X_{\zeta,\delta}$ are not bounded below. Eberlein and Keller suggested overcoming this problem by introducing a stopping time $\tau = \inf\{t > 0; \Delta X_{\zeta,\delta}(t) < -1\}$ and working with $\hat{X}_{\zeta,\delta}$ instead of $X_{\zeta,\delta}$, where for each $t \geq 0$,

$$\hat{X}_{\zeta,\delta}(t) = X_{\zeta,\delta}(t)\chi_{\{t \leq \tau\}},$$

but this is clearly a somewhat contrived approach. An alternative point of view, also put forward by Eberlein and Keller is to model stock prices by an exponential hyperbolic Lévy process and utilise

$$S(t) = S(0)e^{X_{\zeta,\delta}(t)}.$$

As usual we discount and consider

$$\hat{S}(t) = S(0)e^{X_{\zeta,\delta}(t) - rt},$$

and we require a measure Q with respect to which $\hat{S} = (\hat{S}(t), t \geq 0)$ is a martingale. As expected, the market is incomplete, and we use the Essher transform to price the option. Hence we seek a measure of the form Q_u, which satisfies

$$\left.\frac{dQ_u}{dP}\right|_{\mathcal{F}_t} = N_u(t) = \exp\left(-uX_{\zeta,\delta}(t) - t\log(M_{\zeta,\delta}(u))\right).$$

$M_{\zeta,\delta}(u)$ herein denotes the moment generating function of $X_{\zeta,\delta}(1)$, as given by Proposition 3.1, for $|u| < \alpha$. Recalling Lemma 3.2, we see that $\hat{S}$ is a Q-martingale if and only if $\hat{S}N_u = (\hat{S}(t)N_u(t), t \geq 0)$ is a P-martingale. Now

$$\hat{S}(t)N_u(t) = \exp\left((1-u)X_{\zeta,\delta}(t) - t(\log(M_{\zeta,\delta}(u) + r)\right).$$

But we know that $(\exp\left((1-u)X_{\zeta,\delta}(t) - t(\log(M_{\zeta,\delta}(1-u)))\right)), t \geq 0)$ is a martingale and comparing the last two facts, we find that $\hat{S}$ is a Q-martingale if and only if

$$\begin{aligned} r &= \log(M_{\zeta,\delta}(1-u)) - \log(M_{\zeta,\delta}(u)) \\ &= \log\left[\frac{K_1(\sqrt{\zeta^2 - \delta^2(1-u)^2})}{K_1(\sqrt{\zeta^2 - \delta^2 u^2})}\right] - \frac{1}{2}\log\left[\frac{\zeta^2 - \delta^2(1-u)^2}{\zeta^2 - \delta^2 u^2}\right]. \end{aligned}$$

The required value of u can now be determined from this expression by numerical means.

We can now price a European call option with strike price k and expiration time T. Writing $S(0) = s$ as usual, the price is

$$\begin{aligned} V(0) &= \mathbb{E}_{Q_u}(e^{-rT}(S(T) - k)^+) \\ &= \mathbb{E}_{Q_u}(e^{-rT}(se^{X_{\zeta,\delta}(t)} - k)^+). \end{aligned}$$

If we now let $f_{\zeta,\delta}^{(t)}$ be the pdf of $X_{\zeta,\delta}(t)$ with respect to P, we can use the Esscher transform to show that $X_{\zeta,\delta}(t)$ also has a pdf with respect to Q_u which is given by

$$f_{\zeta,\delta}^{(t)}(x; u) = f_{\zeta,\delta}^{(t)}(x)e^{-ux - t\log(M_{\zeta,\delta}(u))},$$

for each $x \in \mathbb{R}, t \geq 0$. We thus obtain the pricing formula

$$V(0) = s\int_{\log(\frac{k}{s})}^{\infty} f_{\zeta,\delta}^{(T)}(x; 1 - u)dx - e^{-rT}k\int_{\log(\frac{k}{s})}^{\infty} f_{\zeta,\delta}^{(T)}(x; u)dx.$$

Finally we discuss the volatility as promised. Suppose that instead of a hyperbolic process, we reverted to a Brownian motion model of logarithmic stock price growth and wrote $S(t) = e^{Z(t)}$ where $Z(t) = \sigma B(t)$ for each $t \geq 0$, then the volatility is given by $\sigma^2 = \mathbb{E}(Z(1)^2)$. By analogy, we define the volatility in the hyperbolic case by $\sigma^2 = \mathbb{E}(X_{\zeta,\delta}(1)^2)$. From the moment generating function and properties of Bessel functions, we obtain

$$\sigma^2 = \frac{\delta^2 K_2(\zeta)}{\zeta K_1(\zeta)}.$$

3.9 The Generalised Black-Scholes Equation

In their original work, Black and Scholes derived a partial differential equation for the price of a European option. It is worth trying to imitate this in the general Lévy market. We price our option using the Esscher transform, to establish that there is a measure Q such that $\tilde{S}(t) = e^{-rt}\mathcal{E}_X(t)$ is a martingale, hence

$$d\tilde{S}(t) = \sigma\tilde{S}(t-)dB_Q(t) + \int_{(c,\infty)} \sigma\tilde{S}(t-)x\tilde{N}_Q(dt, dx).$$

(where we have taken $\kappa = 1$, for convenience). It follows that

$$dS(t) = rS(t-)dt + \sigma S(t-)dB_Q(t) + \int_{(c,\infty)} \sigma S(t-)x\tilde{N}_Q(dt, dx).$$

We consider a contingent claim X and construct an associated function $s \to X(s)$ which describes its dependence on that of the underlying stock. The value of the option at time t is:

$$C(t,s) = \mathbb{E}_Q(e^{-r(T-t)}X(S(t))|S(t)=s).$$

Now consider the integro-partial differential operator $\mathcal{L}$ defined on functions which are twice differentiable in the space variable and differentiable in the time variable:

$$\begin{aligned}(\mathcal{L}F)(t,x) &= \frac{\partial F}{\partial t}(t,x) + rx\frac{\partial F}{\partial x}(t,x) + \frac{\sigma^2 x^2}{2}\frac{\partial^2 F}{\partial x^2}(t,x) - rF(t,x) \\ &+ \int_{(c,\infty)} \left[F(t,x(1+\sigma y)) - F(t,x) - x\sigma y\frac{\partial F}{\partial x}(t,x)\right]\nu(dy)\end{aligned} \tag{3.15}$$

Theorem 3.5 *If $\mathcal{L}F = 0$ with terminal boundary condition $F(T,z) = X(z)$ then $F = C$.*

Proof. First consider the related integro-differential operator

$$\mathcal{L}_0 F(t,x) = \mathcal{L}F(t,x) + rF(t,x).$$

It is an easy exercise in calculus to deduce that $\mathcal{L}F = 0$ if and only if $\mathcal{L}_0 G = 0$, where $G(t,x) = e^{-rt}F(t,x)$. By Itô's formula:

$$\begin{aligned}G(T,S(T)) - G(t,S(t)) &= \text{a } Q-\text{martingale } + \int_t^T \mathcal{L}_0 G(u,S(u-))du \\ &= \text{a } Q-\text{martingale},\end{aligned}$$

hence

$$G(t,s) = \mathbb{E}_Q(G(T,S(T))|S(t)=s)$$

and we thus deduce that

$$\begin{aligned}F(t,s) &= e^{-r(T-t)}\mathbb{E}_Q(F(T,S(T))|S(t)=s) \\ &= e^{-r(T-t)}\mathbb{E}_Q(X(S(T))|S(t)=s),\end{aligned}$$

as was required. □

If we take $\nu = 0$ so we have a stock market driven solely by Brownian motion, then we recapture the famous *Black-Scholes pde*. The more general operator $\mathcal{L}$ is much more complicated, nonetheless both analytic and numerical methods have been devised to enable it to be applied to option pricing problems.

3.10 An invitation to Malliavin calculus

Throughout this section, we will work solely with one-dimensional Brownian motion $B = (B(t), t \geq 0)$. We also take the canonical version of B, i.e. we realise it as $B(t)(\omega) = \omega(t)$ for $\omega \in \Omega$, where Ω is the space of continuous paths from $\mathbb{R}^+$ to $\mathbb{R}$ which vanish at the origin (P is Wiener measure). We should however bear in mind that the development of Malliavin calculus for more general Lévy processes and their financial application is an area where there is currently a great deal of activity, specifically through the work of Bernt Øksendal and his collaborators.

Let's return to the predictable representation of a square-integrable claim X

$$X = \mathbb{E}(X) + \int_0^T F(s)dB(s).$$

It would be very helpful to have more information about the process F. This is given by the *Clark-Ocone formula* which tells us that for sufficiently smooth F

$$F(s) = \mathbb{E}(D_s(X)|\mathcal{F}_s),$$

where D_s is the Malliavin derivative.

Malliavin calculus (also called *the stochastic calculus of variations*) was originally developed by Paul Malliavin at the end of the 1970s as part of his scheme for obtaining a probabilistic technique for determining when the transition densities of solutions of stochastic differential equations are smooth. The three main ingredients of this are:

- **Wiener chaos**

  Multiple stochastic integrals are defined on *symmetric* multilinear square-integrable (deterministic) functions on $[0, T]^n$ by continuous linear extension of the following procedure:

  $$\text{if } f(t_1, \ldots, t_n) = \sum_{i_1, \ldots, i_n = 1}^{p} a_{i_1 \ldots i_n} \chi_{A_1}(t_1) \cdots \chi_{A_n}(t_n),$$

  then the multiple Wiener integral $\int_{[0,T]^n} f(t_1, \ldots, t_n)dB(t_1) \ldots dB(t_n)$ is usually denoted as $I_n(f)$ and is defined by

  $$I_n(f) = \sum_{i_1, \ldots, i_n = 1}^{p} a_{i_1 \ldots i_n} B(A_1) \cdots B(A_n),$$

where $B([s,t]) = B(t) - B(s)$.

These are orthogonal:-

$$\mathbb{E}(I_m(f)I_n(g)) = \begin{cases} 0 & \text{if } m \neq n \\ n!\langle f, g\rangle_{L^2([0,T]^n)} & \text{if } m = n \end{cases}$$

Any $F \in L^2(\Omega, \mathcal{F}, P)$ has a *Wiener chaos expansion*

$$F = \mathbb{E}(F) + \sum_{n=1}^{\infty} I_n(f_n).$$

This should be compared with the predictable representation.

- **The Malliavin Derivative**

  If $F \in L^2(\Omega, \mathcal{F}, P)$ is such that

  $$\sum_{n=1}^{\infty} nn!||f_n||^2_{L^2([0,T]^n)} < \infty,$$

  we can define, for each $t \in (0, T]$

  $$D_t F = \sum_{n=1}^{\infty} nI_{n-1}(f_n(\cdot, t)),$$

  where $I_{n-1}(f_n(\cdot, t))$ signifies that we compute the multiple integral only with respect to the first $n-1$ variables of $t_1, \ldots, t_{n-1}, t$.

- **The Skorohod Integral**

  If $u = (u(t), t \geq 0)$ is a process for which each $u(t) \in L^2(\Omega, \mathcal{F}, P)$, then it has a Wiener chaos expansion, $u(t) = \mathbb{E}(u(t)) + \sum_{n=1}^{\infty} I(f_n(t))$. $\tilde{f}_n$ will denote the extension of f_n to a *symmetric* function of $n+1$ variables (t is the extra dimension). If u is such that

  $$\sum_{n=1}^{\infty}(n+1)!||\tilde{f}_n||^2_{L^2([0,T]^{n+1})} < \infty,$$

  we define its *Skorohod integral*:

  $$\int_0^T u(t)\delta B(t) = \sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_n).$$

We sometimes write $\delta(u) = \int_0^T u(t)\delta B(t)$. δ is called the *divergence*.

Some properties:-

- The Malliavin derivative and Skorohod integral are mutually adjoint, i.e. if u and F are such that these are well-defined, we have the *integration by parts formula*:

$$\mathbb{E}(F\delta(u)) = \mathbb{E}\left(\int_0^T (D_t F)u(t)dt\right).$$

- The Skorohod integral is a non-anticipating extension of the Itô integral, i.e. if u is predictable and square-integrable:

$$\int_0^T u(t)\delta B(t) = \int_0^T u(t)dB(t).$$

Example

$$\int_0^T B(t)\delta B(t) = \int_0^T B(t)dB(t) = \frac{1}{2}(B(T)^2 - T).$$

The same reasoning cannot be applied to $\int_0^T B(T)\delta B(t)$ as $B(T)$ is only $\mathcal{F}(t)$-adapted for $t \geq T$ and hence is not Itô-integrable. However, $B(T)$ has a chaos expansion with

$$f_1 = 1 \quad \text{and} \quad f_n = 0 \quad \text{for} \quad n \geq 2.$$

Since (as a function of two variables) $\tilde{f}_1 = 1$, we have

$$\begin{aligned} \int_0^T B(T)\delta B(t) &= \int_0^T \int_0^T 1 dB(s)dB(t) \\ &= 2\int_0^T \int_0^t dB(s)dB(t) = 2\int_0^T B(t)dB(t) = B(T)^2 - T. \end{aligned}$$

References and Further Reading

These lectures have been broadly based on my recent book:

D.Applebaum *Lévy Processes and Stochastic Calculus*, Cambridge University Press (2004)

and from an earlier course of lectures partly derived from it, which have been separately published as

D.Applebaum, Lévy processes in Euclidean spaces and groups in *Quantum Independent Increment Processes I: From Classical Probability to Quantum Stochastic Calculus, Springer Lecture Notes in Mathematics* , Vol. **1865** M Schurmann, U. Franz (Eds.) 1-99,(2005)

A comprehensive account of the structure and properties of Lévy processes is:

K-I.Sato, *Lévy Processes and Infinite Divisibility*, Cambridge University Press (1999)

A shorter account, from the point of view of the French school, which concentrates on fluctuation theory and potential theory aspects is

J.Bertoin, *Lévy Processes*, Cambridge University Press (1996)

For an insight into the wide range of both theoretical and applied recent work wherein Lévy processes play a role, consult

O.E.Barndorff-Nielsen,T.Mikosch, S.Resnick (eds), *Lévy Processes: Theory and Applications*, Birkhäuser, Basel (2001)

There are many books on stochastic calculus, stochastic integration etc. A very nice comprehensive introduction, with an emphasis on Brownian motion as noise, can be found in

B.Øksendal, *Stochastic Differential Equations* (sixth edition), Springer-Verlag Berlin Heidelberg (2003)

For those who want the theory in full generality (i.e. based on general semimartingales with jumps), the standard text is

P.Protter, *Stochastic Integration and Differential Equations* (second edition), Springer-Verlag, Berlin Heidelberg (2003)

There are a plethora of books and articles on stochastic aspects of option pricing. For a very nice short but rigorous introduction, read

P.Protter, A partial introduction to financial asset pricing theory, *Stochastic Processes and their Applications* **91**, 169-203 (2001)

Two books have now appeared which focus entirely on models with jumps:

R.Cont, P.Tankov, *Financial Modelling with Jump Processes*, Chapman and Hall/CRC (2004)

is extremely comprehensive and also contains a lot of valuable background material on Lévy processes.

W.Schoutens, *Lévy Processes in Finance: Pricing Financial Derivatives*, Wiley (2003)

is shorter and aimed at a wider audience than mathematicians and statisticians.

One of the best places to get an insight into the Malliavin calculus (while keeping technicalities to a minimum) is

B.Øksendal, An introduction to Malliavin calculus with applications to economics, available from http://www.nhh.no/for/dp/1996/wp0396.pdf

Note that a new book dedicated to financial applications of Malliavin calculus has just been published:

P.Malliavin, A.Thalmaier, *Stochastic Calculus of Variations in Mathematical Finance*, Springer-Verlag (2005)

For a large number of preprints on Lévy processes and related topics, as well as mini-proceedings of the 1998 and 2002 Aarhus conferences on Lévy processes go to the MaPhySto website: http://www.maphysto.dk.