MAS350 Measure and Probability: Background

These notes are intended to give some background to MAS350. Most of the material here should be familiar but there may be one or two minor extensions of ideas you have seen before.

1. *Set Theory.*

   Let S be a set and $A, B, C, \ldots$ be subsets.

   A^c is the complement of A in S so that

   $$A^C = \{x \in S; x \notin A\}.$$

   Union $A \cup B = \{x \in S; x \in A \text{ or } x \in B\}$.

   Intersection $A \cap B = \{x \in S; x \in A \text{ and } x \in B\}$.

   Set theoretic difference: $A - B = A \cap B^c$.

   We have finite and infinite unions and intersections so if $A_1, A_2, \ldots, A_n$ are subsets of S.

   $$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n.$$

   $$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n.$$

   We will also need *infinite* unions and intersections. So let (A_n) be a sequence of subsets in S.

   Let $x \in S$. We say that $x \in \bigcup_{i=1}^{\infty} A_i$ if $x \in A_i$ for at least one value of i. We say that $x \in \bigcap_{i=1}^{\infty} A_i$ if $x \in A_i$ for all values of i.

   Note that *de Morgan's laws* hold in this context:

   $$\left(\bigcap_{i=1}^{\infty} A_i\right)^c = \bigcup_{i=1}^{\infty} A_i^c.$$
   $$\left(\bigcup_{i=1}^{\infty} A_i\right)^c = \bigcap_{i=1}^{\infty} A_i^c.$$

   The *Cartesian product* $S \times T$ of sets S and T is

   $$S \times T = \{(s, t); s \in s, t \in T\}.$$

2. *Sets of Numbers*

   - Natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$.
   - Non-negative integers $\mathbb{Z}_+ = \mathbb{N} \cup \{0\} = \{0, 1, 2, 3, \ldots\}$.
   - Integers $\mathbb{Z}$.
   - Rational numbers $\mathbb{Q}$.
   - Real numbers $\mathbb{R}$.
   - Complex numbers $\mathbb{C}$.

   A set X is *countable* if there exists a bijection between $\mathbb{N}$ and X. A set is *uncountable* if it fails to be countable. $\mathbb{N}, \mathbb{Z}_+, \mathbb{Z}$ and $\mathbb{Q}$ are countable. $\mathbb{R}$ and $\mathbb{C}$ are uncountable.

3. *Images of Sets Under Mappings.*

   Suppose that S_1 and S_2 are two sets and that $f : S_1 \to S_2$ is a mapping (or function). Suppose that $A \subseteq S_1$. The *image* of A under f is the set $f(A) \subseteq S_2$ defined by

   $$f(A) = \{y \in S_2; y = f(x) \text{ for some } x \in S_1\}.$$

   If $B \subseteq S_2$ the *inverse image* of B under f is the set $f^{-1}(B) \subseteq S_1$ defined by

   $$f^{-1}(B) = \{x \in S_1; f(x) \in B\}.$$

   Note that $f^{-1}(B)$ makes sense irrespective of whether the mapping f is invertible.

   Key properties are, with $A, A_1, A_2 \subseteq S_1$ and $B, B_1, B_2 \subseteq S_2$:

   $$f(f^{-1}(B)) \subseteq B \quad , \quad A \subseteq f^{-1}(f(A)),$$

   $$f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2),$$

   $$f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2),$$

   $$f^{-1}(A^c) = f^{-1}(A)^c,$$

   $$f(A_1 \cup A_2) = f(A_1) \cup f(A_2),$$

$$f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2),$$

$$A \subseteq B \Rightarrow f^{-1}(A) \subseteq f^{-1}(B).$$

4. *Extended Real Numbers*

   We sometimes find it convenient to pretend that there is a "number" ∞ such that $x < \infty$ for all $x \in [0, \infty)$. Then the extended non-negative real axis is $[0, \infty]$. ∞ is NOT a real number and we find it convenient to extend addition and multiplication to $[0, \infty]$ by defining

   $$\infty + x = x + \infty = \infty,$$

   $$\infty . x = x . \infty = \infty \text{ for } x \neq 0,$$

   $$\infty . 0 = 0 . \infty = 0.$$

   Note that $\infty - \infty$ and $\frac{\infty}{\infty}$ are undefined. We write $\mathbb{R}^* = \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$.

5. *Analysis.*

   - sup and inf. If A is a bounded set of real numbers, we write $\sup(A)$ and $\inf(A)$ for the real numbers that are their least upper bounds and greatest lower bounds (respectively.) If A fails to be bounded above, we write $\sup(A) = \infty$ and if A fails to be bounded below we write $\inf(A) = -\infty$. Note that $\inf(A) = -\sup(-A)$ where $-A = \{-x; x \in A\}$. If $f : S \to \mathbb{R}$ is a mapping, we write $\sup_{x \in S} f(x) = \sup\{f(x); x \in S\}$. A very useful inequality is

     $$\sup_{x \in S} |f(x) + g(x)| \leq \sup_{x \in S} |f(x)| + \sup_{x \in S} |g(x)|.$$

   - Sequences and Limits. Let $(a_n) = (a_1, a_2, a_3, \ldots)$ be a sequence of real numbers. It *converges* to the real number a if given any $\epsilon > 0$ there exists a natural number N so that whenever $n > N$ we have $|a - a_n| < \epsilon$. We then write $a = \lim_{n \to \infty} a_n$.

     A sequence (a_n) which is *monotonic increasing* (i.e. $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$) and *bounded above* (i.e. there exists $K > 0$ so that $a_n \leq K$ for all $n \in \mathbb{N}$) converges to $\sup_{n \in \mathbb{N}} a_n$.

     A sequence (a_n) which is *monotonic decreasing* (i.e. $a_{n+1} \leq a_n$ for all $n \in \mathbb{N}$) and *bounded below* (i.e. there exists $L > 0$ so that $a_n \geq L$ for all $n \in \mathbb{N}$) converges to $\inf_{n \in \mathbb{N}} a_n$.

     A *subsequence* of a sequence (a_n) is itself a sequence of the form (a_{n_k}) where $n_{k_1} < n_{k_2}$ when $k_1 < k_2$.

- Series. If the sequence (s_n) converges to a limit s where $s_n = a_1 + a_2 + \cdots + a_n$ we write $s = \sum_{n=1}^{\infty} a_n$ and call it the *sum of the series*. If each $a_n \geq 0$ then the sequence (s_n) is either convergent to a limit or properly divergent to infinity. In the latter case we write $s = \infty$ and interpret this in the sense of extended real numbers.

- Continuity. A function $f : \mathbb{R} \to \mathbb{R}$ is *continuous* at $a \in \mathbb{R}$ if given any $\epsilon > 0$ there exists $\delta > 0$ so that $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$. Equivalently f is continuous at a if given any sequence (a_n) that converges to a, the sequence $(f(a_n))$ converges to $f(a)$.

  f is a *continuous function* if it is continuous at every $a \in \mathbb{R}$.