Chapter 4

Probability Measures on Compact Lie Groups

4.1 Classes of Probability Measures; Convolution

Let $\mathcal{P}(G)$ be the set of all Borel probability measures defined on an arbitrary Lie group G. We equip $\mathcal{P}(G)$ with the topology of *weak convergence* so if $(\mu_n, n \in \mathbb{N})$ is a sequence of measures in $\mathcal{P}(G)$ and $\mu \in \mathcal{P}(G)$, we say that the sequence converges to μ weakly as $n \to \infty$ if $\lim_{n\to\infty} \int_G f(x)\mu_n(dx) = \int_G f(x)\mu(dx)$ for all $f \in C_b(G)$. In this case we sometimes write $\mu_n \underset{w}{\to} \mu$ as $n \to \infty$.

If $\mu \in$P(G), its reversed measure $\widetilde{\mu} \in \mathcal{P}(G)$ where $\widetilde{\mu}(A) = \mu(A^{-1})$ for all $A \in \mathcal{P}(G)$. We say that $\mu \in \mathcal{P}(G)$ is *symmetric* if $\mu = \widetilde{\mu}$ and *central* (or *conjugate invariant*) if $\mu(gAg^{-1}) = \mu(A)$ for all $g \in G$ and all $A \in \mathcal{B}(G)$. We write $\mathcal{P}_s(G)$ and $\mathcal{P}_c(G)$ to denote the spaces of central and symmetric Borel probability measures defined on G (respectively) and we define $\mathcal{P}_{sc}(G) = \mathcal{P}_c(G) \cap \mathcal{P}_s(G)$.

If we are given a (left or right) Haar measure on G (whicb is always, as usual, assumed to be normalised when G is compact) we define $\mathcal{P}_{ac}(G)$ for the corresponding subset of $\mathcal{P}(G)$ comprising absolutely continuous measures, so $\mu \in \mathcal{P}_{ac}(G)$ if there exists $f_\mu \in L^1(G)$ so that $\mu(A) = \int_A f_\mu(g)dg$ for all $A \in \mathcal{B}(G)$. The Radon-Nikodým derivative f_μ is called the *density* of the measure μ (with respect to the given Haar measure.) If G is compact and $\mu \in \mathcal{P}_{ac}$ then $\mu \in \mathcal{P}_s$ if and only if $f_\mu(g) = f_\mu(g^{-1})$ for almost all $g \in G$ and $\mu \in \mathcal{P}_s$ if and only if $f_\mu(hgh^{-1}) = f_\mu(g)$ for all $h \in G$ and almost all $g \in G$.

To see that $\mathcal{P}(G) \neq \emptyset$ consider the *Dirac mass* δ_g at the point $g \in G$ which is defined for each $A \in \mathcal{B}(G)$ by $\delta_g(A) = \left.\begin{matrix} 1 & \text{if } g \in A \\ 0 & \text{if } g \notin A \end{matrix}\right\}$. Clearly $\delta_g \in \mathcal{P}(G)$ and $\widetilde{\delta_g} = \delta_{g^{-1}}$. We may also form measures in $\mathcal{P}(G)$ by taking convex combinations of distinct Dirac masses. We will consider many more examples as this and the subsequent chapter unfold.

Let $\mu_1, \mu_2 \in \mathcal{P}(G)$. Using the Riesz representation theorem we may assert the existence in $\mathcal{P}(G)$ of the left and right *convolution* products $\mu_1 *_L \mu_2$ and $\mu_1 *_R \mu_2$ which are defined (respectively) for all $f \in C_c(G)$ as follows:

$$\int_G f(g)(\mu_1 *_L \mu_2)(dg) = \int_G \int_G f(gh)\mu_1(dg)\mu_2(dh),$$

$$\int_G f(g)(\mu_1 *_R \mu_2)(dg) = \int_G \int_G f(hg)\mu_1(dg)\mu_2(dh).$$

From now on we will only deal with left convolution and we will write $\mu_1 * \mu_2 := \mu_1 *_L \mu_2$. It can be shown (see e.g. Stromberg) that for all $B \in \mathcal{B}(G)$

$$\begin{aligned}(\mu_1 * \mu_2)(B) &= \int_G \int_G 1_B(gh)\mu_1(dg)\mu_2(dh) \\ &= \int_G \mu_1(Bh^{-1})\mu_2(dh) = \int_G \mu_2(g^{-1}B)\mu_1(dg). \quad (4.1.1)\end{aligned}$$

Convolution is associative, i.e. if $\mu_1, \mu_2, \mu_3 \in \mathcal{P}(G)$ then $(\mu_1 * \mu_2) * \mu_3 = \mu_1 * (\mu_2 * \mu_3)$ and so $(\mathcal{P}(G), *)$ is a semigroup . But note that if G is not abelian then we cannot expect commutativity to hold. Indeed you can easily check that for $g, h \in G, \delta_g * \delta_h = \delta_{gh}$ and so $\delta_g * \delta_h = \delta_h * \delta_g$ if and only if $gh = hg$. In the general case $(\mathcal{P}(G), *)$ is a monoid (i.e. a semigroup with an identity element) since $\mu * \delta_e = \delta_e * \mu$ for all $\mu \in \mathcal{P}(G)$.

If $\mu \in \mathcal{P}_{ac}(G)$ and $\nu \in \mathcal{P}(G)$ we write $f_\mu * \nu := \mu * \nu$ and $\nu * f_\mu := \nu * \mu$. By using Fubini's theorem we easily verify that if Haar measure is right invariant then $f_\mu * \nu \in \mathcal{P}_{ac}(G)$ with density $\int_G f_\mu(gh^{-1})\nu(dh)$ and if Haar measure is left invariant then $\nu * f_\mu \in \mathcal{P}_{ac}(G)$ with density $\int_G f_\mu(g^{-1}h)\nu(dh)$.

The operation $\tilde{\cdot}$ acts as an involution on $(\mathcal{P}(G), *)$. Indeed we have $\tilde{\tilde{\mu}} = \mu$ for all $\mu \in \mathcal{P}(G)$, $\widetilde{\mu_1 * \mu_2} = \widetilde{\mu_2} * \widetilde{\mu_1}$ for all $\mu_1, \mu_2 \in \mathcal{P}(G)$ and $\widetilde{\delta_e} = \delta_e$.

The *support* of $\mu \in \mathcal{P}(G)$, which we denote by $\text{supp}(\mu)$, is the set of all $g \in G$ for which every Borel neighborhood of g has strictly positive μ-measure. It is clear that $\text{supp}(\mu)$ is a closed subset of G. It is shown in Wendel (pp. 925-6) that if μ_1, μ_2 are regular probability measures on G then

$$\text{supp}(\mu_1 * \mu_2) = \text{supp}(\mu_1)\text{supp}(\mu_2), \quad (4.1.2)$$

where if $A, B \in \mathcal{B}(G), AB := \{gh, g \in A, h \in B\}$ (and for later usage $A^2 := AA$.)

Although we won't use it in the sequel, the next result may be of interest.

Proposition 4.1.1. *If G is a compact group then the space $\mathcal{P}(G)$, equipped with the weak topology, is compact.*

Proof. By identifying each $\mu \in \mathcal{P}(G)$ with the linear functional I_μ on $C(G)$ defined by $I_\mu(f) = \int_G f(g)\mu(dg)$ for $f \in C(G)$ we embed $\mathcal{P}(G)$ into the topological dual space $C(G)^*$ and recognise that the weak topology on $\mathcal{P}(G)$ is in fact the restriction of the weak-$*$ topology on $C(G)^*$. By the Banach-Alaoglu theorem, the unit ball in $C(G)^*$ is weak-$*$ compact. However $\mathcal{P}(G)$ is easily verified to be a closed subset of this ball and the result follows. □

Note that the mapping $g \to \delta_g$ is a continuous embedding of G into a closed subspace of $\mathcal{P}(G)$.

We recall that a family of Borel probability measures $(\mu_\alpha \in \mathcal{I})$ defined on some locally compact space X (where $\mathcal{I}$ is some index set) is *tight* if given any $\epsilon > 0$ there exists a compact set K_ϵ such that $\mu_\alpha(K_\epsilon) > 1 - \epsilon$ for all $\alpha \in \mathcal{I}$. If X is itself compact, then it is clear that any family of probability measures is tight (just take $K_\epsilon = X$ for all ϵ.) So on a compact group G by Prohorov's theorem, any family of Borel probability measures $(\mu_\alpha \in \mathcal{I})$ contains a convergent sequence.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A G-valued *random variable* is a measurable function from $(\Omega, \mathcal{F})$ to $(G, \mathcal{B}(G))$. If X is such random variable, its *law* or *distribution* is the measure $\mu_X \in \mathcal{P}(G)$ defined by $\mu_X(B) = P(X^{-1}(B))$ for all $B \in \mathcal{B}(G)$. The product of two random variables X and Y is the random variable XY whose value at $\omega \in \Omega$ is $X(\omega)Y(\omega)$.[1] If X and Y are independent then the law of XY is the convolution $\mu_X * \mu_Y$.

4.2 The Fourier Transform of a Probability Measure

Let $\mathrm{Rep}(G)$ be the set of all continuous unitary representations of G so for each $\pi \in \mathrm{Rep}(G), g \in G, \pi(g)$ acts as a unitary operator on the complex separable Hilbert space V_π. For each $\mu \in \mathcal{P}(G)$, we define its *Fourier transform* or *characteristic function* $\widehat{\mu}(\pi)$ at $\pi \in \mathrm{Rep}(G)$ to be the bounded linear operator on V_π defined as the Bochner integral

$$\widehat{\mu}(\pi) = \int_G \pi(g^{-1})\mu(dg). \tag{4.2.3}$$

Equivalently it may be defined as a Pettis integral to be the unique bounded linear operator for which

$$\langle \widehat{\mu}(\pi)\phi, \psi \rangle = \int_G \langle \pi(g^{-1}\phi, \psi\rangle \mu(dg),$$

for all $\phi, \psi \in V_\pi$.

Note that if μ is absolutely continuous with respect to a given left Haar measure on μ and has density $f \in L^1(G)$ then our definition is such that $\widehat{\mu}(\pi) = \widehat{f}(\pi)$ where $\widehat{f}(\pi)$ is as defined in Chapter 2.[2]

From now until section 4.7, we will take G to be a compact Lie group and restrict π to be an irreducible representation. So (observing our usual convention

[1] If G is abelian then the binary operation in the group is usually written additively.

[2] It is common in the literature to see the alternative definition "$\widehat{\mu}(\pi) = \int_G \pi(g)\mu(dg)$" which is natural for probabilists but which clashes with the analysts convention that we introduced in Chapter 2.

of identifying equivalence classes with representative elements) we will take $\pi \in \widehat{G}$. Then $\widehat{\mu}(\pi)$ is a $d_\pi \times d_\pi$ matrix and both Bochner and Pettis descriptions are equivalent to defining the matrix elements

$$\widehat{\mu}(\pi)_{ij} = \int_G \pi_{ij}(g^{-1})\mu(dg). \tag{4.2.4}$$

for $1 \leq i, j \leq d_\pi$.

Example 1: Dirac Mass. If $\mu = \delta_g$ for some $g \in G$ then it is easily verified that for all $\pi \in \widehat{G}, \widehat{\mu}(\pi) = \pi(g^{-1})$. In particular $\widehat{\delta_e} = I_\pi$.

Example 2: Normalised Haar measure. We again denote this measure by m. It is easy to see that $m \in \mathcal{P}_{sc}(G)$. We have

$$\widehat{m}(\pi) = \begin{cases} 0 & \text{if } \pi \neq \pi_0 \\ 1 & \text{if } \pi = \pi_0 \end{cases}.$$

To see this it is sufficient to observe that for all $\pi \in \widehat{G}, 1 \leq i, j \leq d_\pi, \widehat{m}(\pi)_{ij} = \int_G \pi_{ij}(g^{-1})dg = \langle 1, \pi_{ij}\rangle_{L^2(G)}$ and the result then follows by Peter-Weyl theory.

Example 3: Standard Gaussian Measures.

We recall the discussion of the heat kernel in section 3.1.1. Now fix a parameter $\sigma > 0$ and consider the heat equation:

$$\frac{\partial u}{\partial t} = \sigma \Delta u, \tag{4.2.5}$$

We write the corresponding heat kernel as $k_\sigma \in C^\infty(\mathbb{R}^+, G)$ and for fixed $t \geq 0$ we will write $k_{t,\sigma}(g) := k_\sigma(t, \cdot) \in C^\infty(G)$. Taking $f = 1$ in (3.1.7) we see immediately that $\int_G k_{t,\sigma}(g)dg = 1$ and so $k_{t,\sigma}$ is the density of a measure $\gamma_{t,\sigma} \in \mathcal{P}(G)$ which we call a *standard Gaussian measure* with parameter σ[3] We now compute the Fourier transform. Using the smoothness of $t \to k_{t,\sigma}$ and a similar dominated convergence argument to that used in the proof of Theorem 3.4.2 we deduce that for all $\pi \in \widehat{G}$,

$$\int_G \pi(g^{-1}) \frac{\partial k_{t,\sigma}(g)}{\partial t} dg = \frac{\partial}{\partial t} \int_G \pi(g^{-1}) k_{t,\sigma}(g) dg,$$

and so the mapping $t \to \widehat{k_{t,\sigma}}(\pi)$ is differentiable and taking Fourier transforms of both sides of (4.2.5) yields that

$$\begin{aligned} \frac{\partial \widehat{k_{t,\sigma}}(\pi)}{\partial t} &= \sigma \widehat{\Delta k_{t,\sigma}}(\pi) \\ &= -\sigma \kappa_\pi \widehat{k_{t,\sigma}}(\pi). \end{aligned}$$

[3]If we were to take a strict analogy with the well-known theory in Euclidean space, we would only use the terminology “standard” Gaussian measure for the case where $\sigma t = \frac{1}{2}$.

Since $\widehat{k_{0,\sigma}(\pi)} = \widehat{\delta_e}(\pi) = I_\pi$, we deduce that

$$\widehat{k_{t,\sigma}}(\pi) = e^{-t\sigma\kappa_\pi} I_\pi. \tag{4.2.6}$$

The next theorem summarises some key properties of the Fourier transform:

Theorem 4.2.1. *For all $\mu, \mu_1, \mu_2 \in \mathcal{P}(G), \pi \in \widehat{G}$,*

1. $\widehat{\mu}(\pi_0) = 1$,
2. $\widehat{\mu_1 * \mu_2}(\pi) = \widehat{\mu_2}(\pi)\widehat{\mu_1}(\pi)$,
3. $||\widehat{\mu}(\pi)||_{op} \leq 1$,
4. $\widehat{\widetilde{\mu}}(\pi) = \widehat{\mu}(\pi)^*$.

Proof.

1. is obvious.
2. For all $1 \leq i, j \leq d_\pi$

$$\begin{aligned} \widehat{\mu_1 * \mu_2}(\pi)_{ij} &= \int_G \pi(h^{-1}g^{-1})\mu_1(dg)\mu_2(dh) \\ &= \sum_{k=1}^{d_\pi} \left(\int_G \pi_{ik}(h^{-1})\mu_2(dh)\right)\left(\int_G \pi_{kj}(g^{-1})\mu_1(dg)\right) \\ &= [\widehat{\mu_2}(\pi)\widehat{\mu_1}(\pi)]_{ij}. \end{aligned}$$

3. For all $\phi \in V_\pi$,

$$\begin{aligned} ||\widehat{\mu}(\pi)\phi|| &= \left|\left|\int_G \pi(g^{-1}\phi\mu(dg)\right|\right| \\ &\leq \int_G ||\pi(g^{-1})\phi||\mu(dg) \\ &= ||\phi||. \end{aligned}$$

4.

$$\begin{aligned} \widehat{\widetilde{\mu}}(\pi) &= \int_G \pi(g^{-1})\widetilde{\mu}(dg) \\ &= \int_G \pi(g)\mu(dg) \\ &= \left(\int_G \pi(g^{-1})\mu(dg)\right)^* = \widehat{\mu}(\pi)^*. \qquad \square \end{aligned}$$

Corollary 4.2.1. *The measure $\mu \in \mathcal{P}_s(G)$ if and only if the matrix $\widehat{\mu}(\pi)$ is self-adjoint for all $\pi \in \widehat{G}$.*

Proof. Necessity is immediate from Theorem 4.2.1 (4). For sufficiency its enough to observe that if the self-adjointness condition holds then for all $\pi \in \widehat{G}, \phi, \psi \in V_\pi$,

$$\int_G \langle \pi(g)\phi, \psi \rangle \mu(dg) = \int_G \langle \pi(g)\phi, \psi \rangle \widetilde{\mu}(dg).$$

Using the Peter-Weyl theorem (Theorem 2.2.4) and dominated convergence we can deduce that $\int_G f(g)\mu(dg) = \int_G f(g)\widetilde{\mu}(dg)$ for all $f \in C(G)$ and the result then follows from the Riesz representation theorem. □

The next theorem generalises Theorem 2.4.1

Theorem 4.2.2. *The measure $\mu \in \mathcal{P}_c(G)$ if and only if $\widehat{\mu}(\pi) = c_\pi I_\pi$ where $c_\pi \in \mathbb{C}$, for all $\pi \in \widehat{G}$.*

Proof. Necessity is established by Schur's lemma just as in Theorem 2.4.1. For sufficiency, for each $h \in G$ define $\mu^h \in \mathcal{P}(G)$ by $\mu^h(A) = \mu(hAh^{-1})$ for $A \in \mathcal{B}(G)$. Then arguing as in the proof of Theorem 2.4.1 we obtain for all $\pi \in \widehat{G}$, $\int_G \pi(g^{-1})\mu(dg) = \int_G \pi(g^{-1})\mu^h(dg)$ and so for all $\phi, \psi \in V_\pi$,

$$\int_G \langle \pi(g)\phi, \psi \rangle \mu(dg) = \int_G \langle \pi(g)\phi, \psi \rangle \mu^h(dg).$$

We can now reach our desired conclusion by arguing as in the proof of Corollary 4.2.1 □

Corollary 4.2.2. *The measure $\mu \in \mathcal{P}_{sc}(G)$ if and only if $\widehat{\mu}(\pi) = c_\pi I_\pi$ where $c_\pi \in \mathbb{R}$, for all $\pi \in \widehat{G}$.*

Proof. This follows immediately from Corollary 4.2.1 and Theorem 4.2.2. □

For example we find by (4.2.6) that standard Gaussian measure is both conjugate-invariant and symmetric.

The remaining results in this section were originally due to Kawada and Itô. The first of these establishes the injectivity of the Fourier transform:

Theorem 4.2.3. *Let $\mu_1, \mu_2 \in \mathcal{P}(G)$. Then $\widehat{\mu_1}(\pi) = \widehat{\mu_2}(\pi)$ for all $\pi \in \widehat{G}$ if and only if $\mu_1 = \mu_2$.*

Proof. Sufficiency is immediate. For necessity let $f \in C(G)$ and let $\epsilon > 0$ be arbitrary. By the Peter-Weyl theorem (Theorem 2.2.4) there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{g \in G} \left| f(g) - \sum_{\pi \in \widehat{G}_0} \sum_{i,j=1}^{d_\pi} \alpha_{ij}^{(\pi)} \pi_{ij}(g) \right| \leq \frac{\epsilon}{2},$$

where $\alpha_{ij}^{(\pi)} \in \mathbb{C}$ $(1 \leq i.j \leq d_\pi)$ and $\widehat{G}_0 \subset \widehat{G}$ with $\#\widehat{G}_0 = n_0$. Then for $k = 1, 2$ we find that

$$\left| \int_G f(g)\mu_k(dg) - \sum_{\pi \in \widehat{G}_0} \sum_{i,j=1}^{d_\pi} \alpha_{ij}^{(\pi)} \widehat{\mu}_k(\pi)_{ij} \right| < \frac{\epsilon}{2},$$

but since $\widehat{\mu}_1(\pi)_{ij} = \widehat{\mu}_2(\pi)_{ij}$ for all $1 \leq i.j \leq d_\pi$, we deduce that

$$\left| \int_G f(g)\mu_1(dg) - \int_G f(g)\mu_2(dg) \right| < \epsilon,$$

and the result follows by the fact that ϵ is arbitrary and by use of the Riesz representation theorem. □

Theorem 4.2.4. *Let $\mu_1, \mu_2 \in \mathcal{P}(G)$. Then $\mu_1 * \mu_2 = \mu_2 * \mu_1$ if and only if $\widehat{\mu_1}(\pi)\widehat{\mu_2}(\pi) = \widehat{\mu_2}(\pi)\widehat{\mu_1}(\pi)$ for all $\pi \in \widehat{G}$.*

Proof. Necessity follows immediately from Theorem 4.2.1(2). For sufficiency observe that

$$\begin{aligned}
\widehat{\mu_1 * \mu_2}(\pi) &= \int_G \pi(gh)\mu_1(dg)\mu_2(dh) \\
&= \left(\int_G \pi(g)\mu_1(dg) \right) \left(\int_G \pi(h)\mu_2(dh) \right) \\
&= \widehat{\mu_1}(\pi)\widehat{\mu_2}(\pi) \\
&= \widehat{\mu_2}(\pi)\widehat{\mu_1}(\pi) \\
&= \left(\int_G \pi(h)\mu_2(dh) \right) \left(\int_G \pi(g)\mu_1(dg) \right) \\
&= \widehat{\mu_2 * \mu_1}(\pi),
\end{aligned}$$

and the required result follows by Theorem 4.2.3. □

Theorem 4.2.5. *Let $(\mu_n, n \in \mathbb{N})$ be a sequence of measures in $\mathcal{P}(G)$. Then $\mu_n \underset{w}{\rightarrow} \mu$ as $n \to \infty$ if and only if $\widehat{\mu_n}(\pi)_{ij} \to \widehat{\mu}(\pi)_{ij}$ as $n \to \infty$ for all $1 \leq i, j, \leq d_\pi, \pi \in \widehat{G}$.*

Proof. If $\mu_n \underset{w}{\rightarrow} \mu$ as $n \to \infty$ then

$$\lim_{n\to\infty} \widehat{\mu_n}(\pi)_{ij} = \lim_{n\to\infty} \int_G \pi_{ij}(g^{-1})\mu_n(dg) = \int_G \pi_{ij}(g^{-1})\mu(dg) = \widehat{\mu}(\pi)_{ij}.$$

Conversely, if $\widehat{\mu_n}(\pi)_{ij} \to \widehat{\mu}(\pi)_{ij}$ as $n \to \infty$ for all $1 \leq i, j, \leq d_\pi, \pi \in \widehat{G}$ then using the same notation and a similar argument to that given in the proof of Theorem 4.2.3 we first observe that for any $f \in C(G), \epsilon > 0$ there exists $n_0 \in \mathbb{N}$ so that for all $n \in \mathbb{N}$

$$\left| \int_G f(g)\mu_n(dg) - \sum_{\pi \in \widehat{G}_0} \sum_{i,j=1}^{d_\pi} \alpha_{ij}^{(\pi)} \widehat{\mu}_n(\pi)_{ij} \right| < \frac{\epsilon}{3},$$

and also

$$\left| \int_G f(g)\mu(dg) - \sum_{\pi \in \widehat{G}_0} \sum_{i,j=1}^{d_\pi} \alpha_{ij}^{(\pi)} \widehat{\mu}(\pi)_{ij} \right| < \frac{\epsilon}{3},$$

But we can also find $n_1 \in \mathbb{N}$ so that if $n > n_1$ we have $|\widehat{\mu_n}(\pi)_{ij} - \widehat{\mu}(\pi)_{ij}| < \frac{\epsilon}{3C}$ for all $1 \leq i, j, \leq d_\pi, \pi \in \widehat{G}_{n_0}$ where $C := \sum_{\pi \in \widehat{G}_0} \sum_{i,j=1}^{d_\pi} |\alpha_{ij}^{(\pi)}|$. From these estimates we can deduce that for all $n > n_1$, $\left|\int_G f(g)\mu_n(dg) - \int_G f(g)\mu(dg)\right| < \epsilon$ and this gives the desired weak convergence. □

Then final result of this section gives a compact Lie group version of the celebrated *Lévy convergence theorem* for sequences of probability measures in Euclidean space.

Theorem 4.2.6. *[Kawada,Itô,Lévy convergence theorem] Suppose that $(\mu_n, n \in \mathbb{N})$ is a sequences of measures in $\mathcal{P}(G)$ and that there exists a family of compatible matrices $(Y(\pi), \pi \in \widehat{G})$ so that $\widehat{\mu_n}(\pi)_{ij} \to Y(\pi)_{ij}$ as $n \to \infty$ for all $1 \leq i, j, \leq d_\pi, \pi \in \widehat{G}$. Then there exists $\mu \in \mathcal{P}(G)$ for which $\mu_n \underset{w}{\to} \mu$ as $n \to \infty$ and $\widehat{\mu}(\pi) = Y(\pi)$ for all $\pi \in \widehat{G}$.*

Proof. Let $f \in C(G)$. Once again using (a straightforward variation of) the same notation to that used in the proof of Theorem 4.2.3 we can assert that given any $m \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ so that

$$\sup_{g \in G} \left| f(g) - \sum_{\pi \in \widehat{G}_0} \sum_{i,j=1}^{d_\pi} \alpha_{ij}^{(\pi,m)}(f)\pi_{ij}(g) \right| \leq \frac{1}{2^m},$$

and so for all $n \in \mathbb{N}$

$$\left| \int_G f(g)\mu_n(dg) - \sum_{\pi \in \widehat{G}_0} \sum_{i,j=1}^{d_\pi} \alpha_{ij}^{(\pi,m)}(f)\widehat{\mu}_n(\pi)_{ij} \right| < \frac{1}{2^m}.$$

Now given any $\epsilon > 0$ and choosing n sufficiently large we obtain for such n and arbitrary m that:

$$\left| \int_G f(g)\mu_n(dg) - \sum_{\pi \in \widehat{G}_0} \sum_{i,j=1}^{d_\pi} \alpha_{ij}^{(\pi,m)}(f)Y(\pi)_{ij} \right| < \frac{1}{2^m} + \epsilon.$$

Define $\Gamma_m(f) := \sum_{\pi \in \widehat{G}_0} \sum_{i,j=1}^{d_\pi} \alpha_{ij}^{(\pi,m)}(f)Y(\pi)_{ij}$. Then from the last inequality we deduce that $(\Gamma_m(f), m \in \mathbb{N})$ is a Cauchy sequence and hence convergent to $\Gamma(f) \in \mathbb{C}$. Again from the last inequality we deduce that $\Gamma(f) = \lim_{n\to\infty} \int_G f(g)\mu_n(dg)$ from which it follows that $f \to \Gamma(f)$ is a positive linear functional on $C(G)$ for which $\Gamma(1) = 1$. Hence by the Riesz representation theorem, there exists a probability measure $\mu \in \mathcal{P}(G)$ for which $\Gamma(f) = \int_G f(g)\mu(dg)$ for all $f \in C(G)$ and this gives the required weak convergence. The fact that $\widehat{\mu}(\pi) = Y(\pi)$ for all $\pi \in \widehat{G}$ then follows from Theorem 4.2.3. □

4.3 Lo-Ng Positivity

Let μ be a Borel probability measure defined on a locally compact abelian group G (with group composition written additively). Let $\widehat{G}$ be the (abelian) dual group of characters (equipped with the compact open topology) and denote the neutral element in $\widehat{G}$ as $\widehat{e}$. We denote the action of $\widehat{G}$ on G in the usual way by using $\langle\cdot,\cdot\rangle$. In this case we have $\widehat{\mu}(x) = \int_G \langle x, g\rangle \mu(dg)$ for all $x \in \widehat{G}$. Let $F : \widehat{G} \to \mathbb{C}$. The celebrated *Bochner theorem* gives a necessary and sufficient condition for $F = \widehat{\mu}$ for some $\mu \in \mathcal{P}(G)$ and this is precisely that $F(\widehat{e}) = 1, F$ is continuous at $\widehat{e}$ and F is positive definite, i.e. $\sum_{i,j=1}^n c_i \overline{c_j} F(x_i - x_j) \geq 0$ for all $n \in \mathbb{N}, c_1, \ldots, c_n \in \mathbb{C}$ and all $x_1, \ldots, x_n \in \widehat{G}$. There is no precise analogue of this result if G is a general compact Lie group. However we can find a necessary and sufficient condition for a family of compatible matrices to be the Fourier transform of a finite measure if we introduce a new notion of positivity due to Lo and Ng as we will now demonstrate. To this end let $C : \widehat{G} \to \mathcal{M}(\widehat{G})$ be a compatible mapping. We say that it is *Lo-Ng positive* if the following holds: If $B : \widehat{G} \to \mathcal{M}(\widehat{G})$ is any other compatible mapping for which $\sum_{\pi \in S} d_\pi \mathrm{tr}(\pi(g)B(\pi)) \geq 0$ for all $g \in G$ and all finite subsets S of $\widehat{G}$ then $\sum_{\pi \in S} d_\pi \mathrm{tr}(\pi(g)C(\pi)B(\pi)) \geq 0$ for all $g \in G$ and all finite subsets S of $\widehat{G}$. It is immediate that if C is Lo-Ng positive and $a \geq 0$ then aC is also Lo-Ng positive. The following gives a useful alternative criterion for Lo-Ng positivity:

Lemma 4.3.1. *Let $B : \widehat{G} \to \mathcal{M}(\widehat{G})$ be a compatible mapping for which $\sum_{\pi \in S} d_\pi tr(\pi(g)B(\pi)) \geq 0$ for all $g \in G$ and all finite subsets S of $\widehat{G}$. Then the compatible mapping C is Lo-Ng positive if and only if $\sum_{\pi \in S} d_\pi tr(B(\pi)C(\pi)) \geq 0$ for all finite subsets S of $\widehat{G}$.*

Proof. First suppose that C is indeed Lo-Ng positive. Then $\sum_{\pi \in S} d_\pi \mathrm{tr}(\pi(g)C(\pi)B(\pi)) \geq 0$ for all $g \in G$ and all finite subsets S of $\widehat{G}$ and the required result follows by taking $g = e$. Conversely suppose the given hypothesis holds. By the assumption on B we have $\sum_{\pi \in S} d_\pi \mathrm{tr}(\pi(gh)B(\pi)) \geq 0$ for all $g, h \in G$ and all finite subsets S of $\widehat{G}$. It follows that $\sum_{\pi \in S} d_\pi \mathrm{tr}(\pi(g)(B(\pi)\pi(h))) \geq 0$ for all $g \in G$. Then by the given hypothesis, for all $h \in G$, $\sum_{\pi \in S} d_\pi \mathrm{tr}(\pi(h)C(\pi)B(\pi)) = \sum_{\pi \in S} d_\pi \mathrm{tr}(C(\pi)(B(\pi)\pi(h))) \geq 0$ and Lo-Ng positivity is established. □

Lemma 4.3.1 equips us with the tool to show that the set of all Lo-Ng positive compatible mappings is closed under taking adjoints. To be precise let $C : \widehat{G} \to \mathcal{M}(\widehat{G})$ be a compatible mapping and define its adjoint $C^* : \widehat{G} \to \mathcal{M}(\widehat{G})$ by the prescription $C^*(\pi) := C(\pi)^*$ for all $\pi \in \widehat{G}$.

Lemma 4.3.2. *If C is a Lo-Ng positive compatible mapping then so is C^*.*

Proof. Let $B : \widehat{G} \to \mathcal{M}(\widehat{G})$ be a compatible mapping for which $\sum_{\pi \in S} d_\pi \mathrm{tr}(\pi(g)B(\pi)) \geq 0$ for all $g \in G$ and all finite subsets S of $\widehat{G}$. Then $\sum_{\pi \in S} d_\pi \mathrm{tr}(\pi(g)B(\pi)^*) = \sum_{\pi \in S} d_\pi \mathrm{tr}(B(\pi)^*\pi(g)) = \sum_{\pi \in S} d_\pi \overline{\mathrm{tr}(\pi(g^{-1})B(\pi))} = \sum_{\pi \in S} d_\pi \mathrm{tr}(\pi(g^{-1})B(\pi)) \geq 0$. So by Lemma 4.3.1,

$$\sum_{\pi \in S} d_\pi \mathrm{tr}(C(\pi)^* B(\pi)) = \sum_{\pi \in S} d_\pi \overline{\mathrm{tr}(B(\pi)^* C(\pi))} = \sum_{\pi \in S} d_\pi \mathrm{tr}(B(\pi)^* C(\pi)) \geq 0$$

and the result follows. □

Before we proceed further we state a useful technical lemma

Lemma 4.3.3. *Let $B, C : \widehat{G} \to \mathcal{M}(\widehat{G})$ be compatible mappings and let S, S' be finite subsets of $\widehat{G}$ with $S \subseteq S'$. Then*

$$\int_G \left(\sum_{\pi' \in S'} d_{\pi'} tr(\pi'(g^{-1})B(\pi')) \right) \left(\sum_{\pi \in S} d_\pi tr(\pi(g)C(\pi)) \right) dg = \sum_{\pi \in S} d_\pi tr(B(\pi)C(\pi)). \tag{4.3.7}$$

Proof. Write both traces on the left hand side of (4.3.7) as finite sums and then use the Schur orthogonality relations. □

The next result begins to establish the crucial link between Lo-Ng positivity and the Fourier transform. Let S be a finite subset of $\widehat{G}$ and $C : S \to \mathcal{M}(\widehat{G})$ be compatible. Note that $f_{S,C} \in C(G)$ where for each $g \in G, f_{S,C}(g) := \sum_{\pi \in S} d_\pi \text{tr}(C(\pi)\pi(g))$.

Theorem 4.3.1. *Let S be a finite subset of $\widehat{G}$. The compatible mapping $C : S \to \mathcal{M}(\widehat{G})$ is Lo-Ng positive if and only if $f_{S,C} \geq 0$. In either case we then have*

$$C(\pi) = \widehat{f_{S,C}}(\pi)$$

for all $\pi \in \widehat{G}$.

Proof. First suppose that $f_{S,C} \geq 0$ and suppose also that $B : \widehat{G} \to \mathcal{M}(\widehat{G})$ be a compatible mapping for which $\sum_{\pi \in S'} d_{\pi'} \text{tr}(\pi'(g)B(\pi')) \geq 0$ for all $g \in G$ and all finite subsets S' of $\widehat{G}$. By the hypothesis on $f_{S,C}$ and (4.3.7) it follows that $\sum_{\pi \in R} d_\pi \text{tr}(C(\pi)\pi(g)) \geq 0$ for all finite subsets R of S and so C is Lo-Ng positive by Lemma 4.3.1. Now suppose that C is Lo-Ng positive but $f_{S,C}(g) \notin [0, \infty)$ for some $g \in G$. Then by continuity there exists an open neighbourhood U of g for which $f_{S,C}(h) \notin [0, \infty)$ for all $h \in U$. By Lo-Ng positivity and Lemma 4.3.1, $\sum_{\pi \in S} d_\pi \text{tr}(\pi(g)B(\pi)) \geq 0 \Rightarrow \sum_{\pi \in S} d_\pi \text{tr}(B(\pi)C(\pi)) \geq 0$ and so we have by (4.3.7),

$$\begin{aligned} & \int_{U^c} \left(\sum_{\pi \in S} d_\pi \text{tr}(\pi(g^{-1})B(\pi)) \right) \left(\sum_{\pi \in S} d_\pi \text{tr}(\pi(g)C(\pi)) \right) dg \\ \leq & \int_G \left(\sum_{\pi \in S} d_\pi \text{tr}(\pi(g^{-1})B(\pi)) \right) \left(\sum_{\pi \in S} d_\pi \text{tr}(\pi(g)C(\pi)) \right) dg \\ = & \sum_{\pi \in S} d_\pi \text{tr}(B(\pi)C(\pi)) \end{aligned}$$

and every term in the last array is non-negative. Consequently we deduce that

$$\begin{aligned}
& \int_U \left(\sum_{\pi\in S} d_\pi \mathrm{tr}(\pi(g^{-1})B(\pi))\right) f_{S,C}(g)dg \\
= & \int_G \left(\sum_{\pi\in S} d_\pi \mathrm{tr}(\pi(g^{-1})B(\pi))\right)\left(\sum_{\pi\in S} d_\pi \mathrm{tr}(\pi(g)C(\pi))\right) dg \\
- & \int_{U^c} \left(\sum_{\pi\in S} d_\pi \mathrm{tr}(\pi(g^{-1})B(\pi))\right)\left(\sum_{\pi\in S} d_\pi \mathrm{tr}(\pi(g)C(\pi))\right) dg \geq 0.
\end{aligned}$$

and this yields a contradiction. Hence we conclude that $f_{S,C}(g) \geq 0$ for all $g \in G$. The last result follows by uniqueness of the Fourier transform. □

Next we state another technical lemma:

Lemma 4.3.4. *There exists a sequence $(\psi_n, n \in \mathbb{N})$ of continuous non-negative functions on G where each $\psi_n(g) = \sum_{\pi\in S_n} d_\pi z_\pi^{(n)} \chi_\pi(g)$ where S_n is a finite subset of $\widehat{G}$ and $z_\pi^{(n)} \in \mathbb{C}$ for all $\pi \in S_n, n \in \mathbb{N}$. This sequence has the following properties:*

(i) $\int_G \psi_n(g)dg = 1$ *for all* $n \in \mathbb{N}$,

(ii) *Given any neighbourhood U of e and any $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\psi_n(g) < \epsilon$ for all $g \in U^c$ and all $n \geq n_0$,*

(iii) $\lim_{n\to\infty} z_\pi^{(n)} = 1$ *for all* $\pi \in \widehat{G}$.

Proof. We follow Talman for (i) and (ii) and Lo-Ng for (iii).

(i) First note that if π is a finite dimensional representation of G then by (3.3.12) we can easily deduce that

$$\sup_{g\in G} |\chi_\pi(g)| \leq d_\pi.$$

Next observe that since G is a compact Lie group, it has a faithful finite dimensional representation π and for all $g, h \in G$,

$$\sum_{i,j=1}^{d_\pi} |\pi_{ij}(g) - \pi_{ij}(h)|^2 > 0,$$

indeed if there is equality, we can easily deduce that π fails to be injective.

Now

$$\begin{aligned}
&\sum_{i,j=1}^{d_\pi} |\pi_{ij}(g) - \pi_{ij}(h)|^2 \\
=\ & \sum_{i,j=1}^{d_\pi} (\pi_{ij}(g) - \pi_{ij}(h))(\overline{\pi_{ij}(g) - \pi_{ij}(h)}) \\
=\ & \sum_{i,j=1}^{d_\pi} (\pi_{ij}(g) - \pi_{ij}(h))(\pi_{ji}(g^{-1}) - \pi_{ji}(h^{-1})) \\
=\ & 2\chi_\pi(e) - \chi_\pi(gh^{-1}) - \overline{\chi_\pi(gh^{-1})}.
\end{aligned}$$

Let $\pi' := \pi \oplus \overline{\pi}$. Then for all $g \in G, \chi_{\pi'}(g) = \chi_\pi(g) + \overline{\chi_\pi(g)}$ and we deduce from the last display that

$$\sup_{g \in G} \chi_{\pi'}(g) < \chi_{\pi'}(e) = d_{\pi'}.$$

Incorporating this with our earlier estimate we see that

$$-d_{\pi'} \leq \sup_{g \in G} \chi_{\pi'}(g) < d_{\pi'}.$$

Now define a new representation π'' of G to be the direct sum of π' and $d_{\pi'}$ copies of the trivial representation. Then for all $g \in G, \chi_{\pi''}(g) = d_{\pi'} + \chi_{\pi'}(g)$ and the estimate we've just established becomes

$$0 \leq \sup_{g \in G} \chi_{\pi''}(g) < 2d_{\pi'} = d_{\pi''}.$$

Now for each $n \in \mathbb{N}, g \in G$ define $\psi_n(g) := c_n \chi_{\pi''}(g)^n$, where $c_n := \int_G \chi_{\pi''}(g)^n dg$. Then by construction ψ_n is continuous, non-negative and $\int_G \psi_n(g)dg = 1$. By Theorem 2.4.2 (ii), $\chi_{\pi''}(g)^n$ is the value at g of the character of the n-fold tensor product of π'' and so by Theorem 2.4.2 (iii), $\chi_{\pi''}(g)^n = \sum_{\pi \in S_n} m_\pi^{(n)} \chi_\pi$. Hence the complex numbers $z_\pi^{(n)}$ appearing in the statement of the lemma are given by $z_\pi^{(n)} = \dfrac{c_n m_\pi^{(n)}}{d_\pi}$.

(ii) For simplicity we write $\chi := \chi_{\pi''}$ and $d := d_{\pi''}$ for the remainder of this proof. Let U be an open neighbourhood of e. Then $G - U$ is compact and so there exists $g_0 \in G - U$ for which $\chi(g_0) = \sup_{g \in G-U} \chi(g)$ and we have $\chi(g_0) < d$. By continuity of $g \to \chi(g)$ at $g = e$, given any $\varepsilon > 0$ there exists an open neighbourhood V of e so that if $g \in V$ then $d - \varepsilon < \chi(g) < d + \varepsilon$. Now choose $\varepsilon = \frac{d - \chi(g_0)}{2}$ and we see that for all $g \in G, \chi(g) > \frac{d + \chi(g_0)}{2}$. Consequently, for each $n \in \mathbb{N}$,

$$\int_V \chi(g)^n dg > m(V) \left(\frac{d + \chi(g_0)}{2} \right)^n.$$

Now

$$\begin{aligned} c_n &< \left(\int_V \chi(g)^n dg\right)^{-1} \\ &< \frac{1}{m(V)}\left(\frac{2}{d+\chi(g_0)}\right)^n. \end{aligned}$$

Then for all $g \in G - U$ we have

$$\begin{aligned} \psi_n(g) &< \frac{1}{m(V)}\left(\frac{2\chi(g)}{d+\chi(g_0)}\right)^n \\ &\leq \frac{1}{m(V)}\left(\frac{2\chi(g_0)}{d+\chi(g_0)}\right)^n, \end{aligned}$$

and we can make the quantity on the right hand side arbitrarily small by taking n to be sufficiently large.

(iii) First note that by (i) and (ii), for given U and ϵ we have for $n \geq n_0$,

$$\left|\int_U \psi_n(g)dg - 1\right| = \left|\int_{U^c} \psi_n(g)dg\right| < \epsilon.$$

If we take the inner product in $L^2(G)$ of ψ_n with a matrix entry of an arbitrary $\pi' \in \widehat{G}$ and use the Peter-Weyl theorem we can easily deduce that for each $n \in \mathbb{N}, \pi \in S_n, z_\pi^{(n)} = \int_G \psi_n(g)\pi_{ij}(g^{-1})dg$ for some $1 \leq i, j \leq d_\pi$. Finally we find that

$$\begin{aligned} \left|z_\pi^{(n)} - \int_U \psi_n(g)dg\right| &= \left|\int_G \psi_n(g)\pi_{ij}(g^{-1})dg - \int_U \psi_n(g)dg\right| \\ &\leq \left|\int_{U^c} \psi_n(g)\pi_{ij}(g^{-1})dg\right| + \left|\int_U (\pi_{ij}(g^{-1}) - 1)\psi_n(g)dg\right| \\ &\leq \int_{U^c} \psi_n(g)dg + \int_U |\pi_{ij}(g^{-1}) - 1|\psi_n(g)dg, \end{aligned}$$

and the required result follows by taking n sufficiently large, U sufficiently small and using the fact that $g \to \pi_{ij}(g^{-1})$ is continuous and takes the value 1 at e. □

The next result is the main one of this section.

Theorem 4.3.2. *[The Lo-Ng Criterion] Let $C : \widehat{G} \to \mathcal{M}(\widehat{G})$ be compatible. Then $C(\pi) = \widehat{\mu}(\pi)$ for all $\pi \in \widehat{G}$ where $\mu \in \mathcal{P}(G)$ if and only if C is Lo-Ng positive with $C(\pi_0) = 1$. Furthermore μ is the weak limit of a sequence $(\mu_n, n \in \mathbb{N})$ where for each $n \in \mathbb{N}, \mu_n \in \mathcal{P}(G)$ is absolutely continuous with respect to Haar measure and has Radon-Nikodým derivative $f_n(g) = \sum_{\pi \in S_n} d_\pi tr(\pi(g)C(\pi))$ for all $g \in G$ where $\#S(m) < \#S(n) < \infty$ if $m < n$.*

Proof Assume that $\mu \in \mathcal{P}(G)$ and $\sum_{\pi \in S} d_\pi \mathrm{tr}(\pi(g)B_\pi) \geq 0$ for all $g \in G$ and all finite subsets S of $\widehat{G}$. Then

$$\sum_{\pi \in S} d_\pi \mathrm{tr}(\widehat{\mu}(\pi)B_\pi) = \int_G \sum_{\pi \in S} d_\pi \mathrm{tr}(\pi(g)B_\pi)\mu(dg) \geq 0$$

and so $\widehat{\mu} : \widehat{G} \to \mathcal{M}(\widehat{G})$ is Lo-Ng positive.

Conversely (and using the notation of Lemma 4.3.4) we have that for all $n \in \mathbb{N}, \sum_{\pi \in S_n} d_\pi \mathrm{tr}(\pi(g)[z_\pi^{(n)} I_{d_\pi}]) \geq 0$ by Lemma 4.3.4 and so since C is assumed to be Lo-Ng positive, we see that $\sum_{\pi \in S_n} d_\pi \mathrm{tr}(\pi(g) z_\pi^{(n)} C(\pi)) \geq 0$. By Lemma 4.3.1 we deduce that the compatible mapping whose value at $\pi \in S_n$ is $z_\pi^{(n)} C(\pi)$ is also Lo-Ng positive. By Theorem 4.3.1, $z_\pi^{(n)} C(\pi) = \int_G \pi(g) h_n(g)$ where for all $g \in G, h_n(g) = \sum_{\pi \in S_n} d_\pi z_\pi^{(n)} \mathrm{tr}(\pi(g)C(\pi))$. Since h_n is continuous it is integrable and as h_n is non-negative, we can define a Borel measure μ_n on G whose Radon-Nikodým derivative is h_n. Using Peter-Weyl theory we have

$$\begin{aligned} \mu_n(G) &= \int_G h_n(g)dg \\ &= \int_G h_n(g)\pi_0(g)dg \\ &= z_{\pi_0}^{(n)} C(\pi_0) = 1. \end{aligned}$$

The fact that $z_{\pi_0}^{(n)} = 1$ follows from Lemma 4.3.4 (i) and the formula $z_\pi^{(n)} = \int_G \psi_n(g)\pi_{ij}(g^{-1})dg$ that is established within the proof of that same lemma. By Prokhorov's theorem, we can now assert that there is a subsequence $(\mu_{n_k}, k \in \mathbb{N})$ that converges weakly to a probability measure μ. By Theorem 4.2.5, we have $\lim_{k\to\infty} \widehat{\mu_{n_k}}(\pi) = \widehat{\mu}(\pi)$ for all $\pi \in \widehat{G}$. But by construction $\lim_{k\to\infty} \widehat{\mu_{n_k}}(\pi) = \lim_{k\to\infty} z_\pi^{(n_k)} C(\pi) = C(\pi)$ by Lemma 4.3.4 (iii). Hence the converse is established.

To prove the last part of the theorem let $h \in C(G)$. Then by the Peter-Weyl theorem, there exists a sequence of matrices $(H_n, n \in \mathbb{N})$ where each H_n acts in a finite dimensional complex Hilbert space of dimension d_n such that $h(g) = \lim_{n\to\infty} \sum_{i=1}^n d_i \mathrm{tr}(\pi_i(g)^* H_i)$ and the convergence is uniform in $g \in G$.

Using Schur orthogonality and (4.3.7) we find that

$$\begin{aligned}
& \int_G h(g)\left(\sum_{\pi\in S_n} d_\pi \mathrm{tr}(\pi(g)C(\pi))\right) dg \\
= & \lim_{m\to\infty}\int_G \left(\sum_{i=1}^m d_i \mathrm{tr}(\pi_i(g)^* H_i)\right)\left(\sum_{\pi\in S_n} d_\pi \mathrm{tr}(\pi(g)C(\pi))\right) dg \\
= & \sum_{\pi\in S_n} d_\pi \mathrm{tr}(H(\pi)C(\pi)) \\
= & \int_G \left(\sum_{\pi\in S_n} d_\pi \mathrm{tr}(\pi(g)^* H(\pi))\right) dg \\
\to & \int_G h(g)\mu(dg),
\end{aligned}$$

as $n \to \infty$ using the dominated convergence theorem.

Remarks

1. Although Lo-Ng positivity is an interesting theoretical result, it seems very difficult to use in practice to determine whether a given family of compatible matrices really is the Fourier transform of a finite measure.

2. As positive-definiteness (in the usual sense) is a key component of Bochner's theorem on locally compact abelian groups, it is worth pointing out that there is a general notion of positive definiteness for functions on a more general locally compact group G. Indeed a continuous function $f : G \to \mathbb{C}$ is positive definite if and only if $\sum_{i,j=1}^n c_i\overline{c_j} f(g_i g_j^{-1}) \geq 0$ for all $g_1, \ldots, g_n \in G, c_1, \ldots, c_n \in \mathbb{C}, n \in \mathbb{N}$. You can learn about these functions in e.g. section 2.8 of Edwards or section 32 of Hewitt and Ross. Note that there is even a *Bochner theorem* which describes the structure of such functions as linear combinations of certain elementary ones, but readers should be warned that it is not related to the Bochner theorem that we have been discussing (i.e. it does not give information about Fourier transforms of finite measures.)

4.4 Absolute Continuity

We investigate absolute continuity of probability measures on G with respect to normalised Haar measure m. We follow the account in Wehn.

Theorem 4.4.1. *[Raikov-Williamson] Let $\mu \in \mathcal{P}(G)$. Then $\mu \in \mathcal{P}_{ac}(G)$ if and only if either $\mu(Eg) \to \mu(E)$ or $\mu(gE) \to \mu(E)$ as $g \to e$ for all $E \in \mathcal{B}(G)$.*

Proof. We only deal here with the case $\mu(Eg) \to \mu(E)$ as $g \to e$. The other limit is dealt with by a similar argument.

First assume that $\mu << m$ and let $f_\mu := \frac{d\mu}{dm}$. Then for all $E \in \mathcal{B}(G)$,

$$\begin{aligned} |\mu(Eg) - \mu(E)| &\leq \int_E |f_\mu(hg^{-1}) - f_\mu(h)|dh \\ &\leq ||R_{g^{-1}}f_\mu - f_\mu||_1 \to 0 \text{ as } g \to e, \end{aligned}$$

by Proposition 1.2.1. Conversely suppose that $\mu(Eg^{-1}) \to \mu(E)$ as $g \to e$ and suppose that $E \in \mathcal{B}(G)$ exists with $m(E) = 0$ and $\mu(E) > 0$. We seek a contradiction. Let $\rho \in L^1(G)$ be such that $\rho \geq 0$ and $\int_G \rho(g)dg = 1$. Then we may define a measure $\nu_\rho \in \mathcal{P}_{ac}(G)$ by $\nu_\rho(A) = \int_A \rho(g)dg$ for all $A \in \mathcal{B}(G)$. Then for all $g \in G$,

$$\begin{aligned} \nu_\rho(g^{-1}E) &= \int_G \rho(h)1_E(gh)dh \\ &= \int_G \rho(g^{-1}h)1_E(h)dh = 0, \end{aligned}$$

since $m(E) = 0$. Hence by (4.1.1)

$$(\mu * \nu_\rho)(E) = \int_G \nu_\rho(g^{-1}E)\mu(dg) = 0.$$

But again by (4.1.1) we have

$$(\mu * \nu_\rho)(E) = \int_G \mu(Eg^{-1})\nu_\rho(dg) > 0,$$

and this yields the required contradiction. □

For each $\mu \in \mathcal{P}(G)$ we define the associated *convolution operator* $T_\mu : B_b(G) \to B_b(G)$ by

$$(T_\mu f)(\sigma) := (f * \mu)(\sigma) = \int_G f(\sigma\tau)\mu(d\tau),$$

for all $f \in B_b(G), \sigma \in G$. It is easy to see that T_μ is linear and a contraction. Furthermore if $\mu, \nu \in \mathcal{P}(G)$ we have

$$T_{\mu*\nu} = T_\mu T_\nu. \tag{4.4.8}$$

It is an important fact that $T_\mu : C(G) \to C(G)$. To see this, let $\sigma_1, \sigma_2 \in G$ and observe that for all $f \in C(G)$,

$$|T_\mu f(\sigma_1) - T_\mu f(\sigma_2)| \leq \int_G |f(\sigma_1\tau) - f(\sigma_2\tau)|\mu(d\tau) = ||L_{\sigma_1^{-1}}f - L_{\sigma_2^{-1}}f||_1,$$

and the result follows by Proposition 1.2.1.

Before we state and prove the next result we recall that a subset S of $C(G)$ is *equicontinuous* if given any $\epsilon > 0$, each $g \in G$ has an open neighbourhood U_g so that if $h \in U_g$ then $|f(g) - f(h)| < \epsilon$ for all $f \in S$.

Theorem 4.4.2. *[Raikov] Let $\mu \in \mathcal{P}(G)$. hen $\mu \in \mathcal{P}_{ac}(G)$ if and only if $T_\mu : C(G) \to C(G)$ is compact.*

Proof. First suppose that $\mu << m$ and write $\rho_\mu := \dfrac{d\mu}{dm}$. Let $(f_n, n \in \mathbb{N})$ be a bounded sequence in $C(G)$. Then for all $g, h \in G, n \in \mathbb{N}$,

$$\begin{aligned} T_\mu f_n(g) - T_\mu f_n(h) &= \int_G f_n(g\tau)\rho_\mu(\tau)d\tau - \int_G f_n(h\tau)\rho_\mu(\tau)d\tau \\ &= \int_G f_n(\tau)(\rho_\mu(g^{-1}\tau) - \rho_\mu(h^{-1}\tau))d\tau, \end{aligned}$$

from which we easily deduce that

$$|T_\mu f_n(g) - T_\mu f_n(h)| \leq \sup_{n\in\mathbb{N}} ||f_n||_\infty ||L_g\rho_\mu - L_h\rho_\mu||.$$

Then equicontinuity of $\{T_\mu f_n, n \in \mathbb{N}\}$ follows from Proposition 1.2.1. We can now appeal to the Arzela-Ascoli theorem to deduce that $\{T_\mu f_n, n \in \mathbb{N}\}$ is relatively compact and so it contains a convergent subsequence. It follows that T_μ is compact.

Conversely suppose that T_μ is compact and let E be an open set in G. Then we can find a sequence $(f_n, n \in \mathbb{N})$ in $C(G)$ which increases monotonically to 1_E. So in particular this sequence is bounded. Hence by assumption, its image contains a convergent subsequence $(T_\mu f_{n_k}, k \in \mathbb{N})$ and for all $g \in G$,

$$\lim_{k\to\infty} T_\mu f_{n_k}(g) = T_\mu 1_E(g) = \mu(g^{-1}E).$$

Furthermore the mapping $g \to \mu(g^{-1}E)$ is continuous and so $\mu(g^{-1}E) \to \mu(E)$ as $g \to e$. By regularity of μ the same limiting behaviour holds for $E \in \mathcal{B}(G)$ and so by Theorem 4.4.1 we deduce that $\mu << m$, as required. □

In the case $G = \Pi^1$, the celebrated theorem of F. and M. Riesz gives a sufficient condition for a probability measure μ to be absolutely continuous. In that case $\widehat{G} = \mathbb{Z}$ and $\widehat{\mu}(n) = \frac{1}{2\pi}\int_0^{2\pi} e^{-inx}\mu(dx)$ for each $n \in \mathbb{Z}$. The required condition for absolute continuity is that $\widehat{\mu}(n) = 0$ for all $n < 0$ (see e.g. Rudin's book or Katznelson.) This result has been extended to compact Lie groups by Brummelhuis (see also his earlier paper.) For ease of exposition, we state it here in the case where G is also connected and semisimple. Let $\pi \in \widehat{G}$ and recall that $V_\pi = \bigoplus_{\mu\in\mathcal{W}(\pi)} V_\mu$ where $\mathcal{W}(\pi)$ is the set of weights of π. Let λ be the highest weight and define $V_\pi^0 := V_\pi \ominus V_\lambda$.

Theorem 4.4.3 (Brummelhuis)**.** *Let G is a compact, connected, semisimple Lie group. If $\mu \in \mathcal{P}(G)$ is such that $\widehat{\mu}(\pi)v = 0$ for all $v \in V_\pi^0$ and for all $\pi \in \widehat{G}$ then $\mu << m$.*

4.5 Regularity of Densities

In this section, we will investigate conditions for a probability measure on a compact group to have a square-integrable, continuous and smooth density of various orders. Although this topic is closely related to that of the previous section, we will make no use of the results that we obtained there.

In this section we first examine the case where $\mu \in \mathcal{P}(G)$ has a square-integrable density.

Theorem 4.5.1. *Let G be a compact Lie group. Then $\mu \in \mathcal{P}(G)$ has an L^2-density f_μ if and only if*

$$\sum_{\pi \in \widehat{G}} d_\pi ||\widehat{\mu}(\pi)||_{HS}^2 < \infty.$$

In this case

$$f_\mu(\sigma) = \sum_{\pi \in \widehat{G}} d_\pi tr(\widehat{\mu}(\pi)\pi(\sigma)) \ a.e.. \tag{4.5.9}$$

Proof. Necessity is straightforward. For sufficiency define $f_\mu := \sum_{\pi \in \widehat{G}} d_\pi \mathrm{tr}(\widehat{\mu}(\pi)\pi)$. Then $f_\mu \in L^2(G, \mathbb{C})$ since $||f_\mu||_2^2 = \sum_{\pi \in \widehat{G}} d_\pi ||\widehat{\mu}(\pi)||_{HS}^2 < \infty$, and by uniqueness of Fourier coefficients $\widehat{f_\mu}(\pi) = \widehat{\mu}(\pi)$ for all $\pi \in \widehat{G}$. Recall that $\mathcal{E}(G)$, which is the set of all continuous functions on G that have only finitely many non-zero Fourier coefficients, is norm dense in $C(G, \mathbb{C})$. Let $h \in \mathcal{E}(G)$. Then there exists a finite subset S of $\widehat{G}$ so that $h(\sigma) = \sum_{\pi \in \widehat{G}} d_\pi \mathrm{tr}(\widehat{h}(\pi)\pi(\sigma)$ for all $\sigma \in G$. Furthermore, by the Schur orthogonality relations, $\widehat{h}(\pi) = 0$ if $\pi \in S^c$. Using the Plancherel-Parseval identity, for each $h \in \mathcal{E}(G)$:

$$\begin{aligned}
\int_G h(\sigma)\overline{f_\mu(\sigma)}d\sigma &= \sum_{\pi \in \widehat{G}} d_\pi \mathrm{tr}(\widehat{h}(\pi)\widehat{\mu}(\pi)^*) \\
&= \sum_{\pi \in S} d_\pi \mathrm{tr}(\widehat{h}(\pi)\widehat{\mu}(\pi)^*) \\
&= \int_G \sum_{\pi \in S} d_\pi \mathrm{tr}(\widehat{h}(\pi)\pi(\sigma))\mu(d\sigma) \\
&= \int_G h(\sigma)\mu(d\sigma).
\end{aligned}$$

By a standard density argument, it then follows that $\int_G h(\sigma)\overline{f_\mu(\sigma)}d\sigma = \int_G h(\sigma)\mu(d\sigma)$, for all $h \in C(G, \mathbb{C})$. The Riesz representation theorem implies that f_μ is real valued and $f_\mu(\sigma)d\sigma = \mu(d\sigma)$. The fact that f_μ is non-negative then follows from the Jordan decomposition for signed measures. □

Note that we can also write (4.5.9) as

$$f_\mu(\sigma) = 1 + \sum_{\pi \in \widehat{G} - \{\pi_0\}} d_\pi \mathrm{tr}(\widehat{\mu}(\pi)\pi(\sigma)) \text{ a.e..}$$

Next we examine continuity of densities:

Proposition 4.5.1. *Let $\mu \in \mathcal{P}(G)$. A sufficient condition for μ to have a continuous density f_μ is that the infinite series $\sum_{\pi \in \widehat{G}} d_\pi tr(\widehat{\mu}(\pi)\pi(\sigma))$ converges uniformly in $\sigma \in G$.*

Proof. Define $f_\mu(\sigma) = \sum_{\pi \in \widehat{G}} d_\pi \text{tr}(\widehat{\mu}(\pi)\pi(\sigma))$ for all $\sigma \in G$. Then $f_\mu \in C(G, \mathbb{C})$ and by uniqueness of Fourier coefficients $\widehat{f_\mu}(\pi) = \widehat{\mu}(\pi)$ for all $\pi \in \widehat{G}$. We now argue as in the proof of Theorem 4.5.1. □

More concrete sufficient conditions for μ to have a continuous density are as follows. In the second of these, for each $\mu \in \mathcal{P}(G)$ we employ the notation $\widehat{\mu}(\lambda) := \widehat{\mu}(\pi_\lambda)$ where $\lambda \in D$ is the highest weight corresponding to $\pi_\lambda \in \widehat{G}$:

- $\sum_{\pi \in \widehat{G}} d_\pi^{\frac{3}{2}} ||\widehat{\mu}(\pi)||_{HS} < \infty$,
- $||\widehat{\mu}(\pi_\lambda)||_{HS} = O(|\lambda|^{-s})$ as $|\lambda| \to \infty$, where $s > r + \frac{m}{2}$.

The first of these is implicit in the first part of the proof of Proposition 3.3.2 (see also Faraut, pp.117-9.) and the second is a direct consequence of the statement of Proposition 3.3.2.

Next we investigate differentiability of densities. Recall that $\{\kappa_\pi, \pi \in \widehat{G}\}$ is the Casimir spectrum of G.

Theorem 4.5.2. *If $\mu \in \mathcal{P}(G)$ and there exists $p \in \mathbb{N}$ so that*

$$\sum_{\pi \in \widehat{G}} d_\pi (1 + \kappa_\pi)^p ||\widehat{\mu}||_{HS}^2 < \infty,$$

then μ has a C^k density for all $k < p - \frac{d}{2}$.

Proof. Since $\kappa_\pi \geq 0$ for all $\pi \in \widehat{G}$, we have $\sum_{\pi \in \widehat{G}} d_\pi ||\widehat{\mu}||_{HS}^2 < \infty$ and so by Theorem 4.5.1, μ has a L^2-density f_μ and $\widehat{f_\mu}(\pi) = \widehat{\mu}(\pi)$ for all $\pi \in \widehat{G}$. The result then follows by Proposition 3.1.4. and the Sobolev embedding theorem (Theorem 3.1.3). □

The next result establishes necessary and sufficient conditions for densities to exist and to be C^∞. Recall that $\mathcal{S}(D)$ is Sugiura space as was introduced in section 3.4.

Theorem 4.5.3. *Let G be a compact connected Lie group. $\mu \in \mathcal{P}(G)$ has a C^∞ density if and only if $\widehat{\mu} \in \mathcal{S}(D)$.*

Proof. Necessity is obvious. For sufficiency its enough to show that μ has an L^2-density. Choose $s > r$ so that Suguira's zeta function converges. Then using Theorem 4.5.1 we have

$$\begin{aligned}
\sum_{\lambda \in D - \{0\}} d_\lambda ||\widehat{\mu}_\lambda||_{HS}^2 &\leq N \sum_{\lambda \in D - \{0\}} |\lambda|^m ||\widehat{\mu}_\lambda||_{HS}^2 \\
&\leq N \sup_{\lambda \in D - \{0\}} |\lambda|^{m+s} ||\widehat{\mu}_\lambda||_{HS}^2 \sum_{\lambda \in D - \{0\}} \frac{1}{|\lambda|^s} \\
&< \infty. \quad \square
\end{aligned}$$

The next result gives an application of Theorem 4.5.3. First we note a useful and easily verified inequality for matrices. If $A, B \in M_n(\mathbb{C})$ then

$$||AB||_{HS} \leq ||A||_{\text{op}}||B||_{HS} \text{ and } ||AB||_{HS} \leq ||B||_{\text{op}}||A||_{HS} \tag{4.5.10}$$

Corollary 4.5.1. *Let G be a compact connected Lie group. Let $\mu \in \mathcal{P}(G)$ be arbitrary and $\gamma_{t,\sigma}$ be a standard Gaussian measure with parameters $t, \sigma > 0$. Then the measures $\mu * \gamma_{t,\sigma}$ and $\gamma_{t,\sigma} * \mu$ have smooth densities.*

Proof. Its sufficient to establish the result for $\mu * \gamma_{t,\sigma}$. First note that by Theorem 4.2.1, (4.5.10) and (4.2.6) for all $\lambda \in D$,

$$||\widehat{\mu * \gamma_{t,\sigma}}(\lambda)||_{HS} = ||\widehat{\gamma_{t,\sigma}}(\lambda)\widehat{\mu}(\lambda)||_{HS} \leq ||\widehat{\mu}(\lambda)||_{\text{op}}||\widehat{\gamma_{t,\sigma}}(\lambda)||_{HS} \leq d_\lambda e^{-t\sigma\kappa_\lambda}.$$

But then using the dimension estimate of Corollary 2.5.2 and (2.5.15) we obtain for all $p \in \mathbb{N}$,

$$\limsup_{|\lambda|\to\infty} |\lambda|^p ||\widehat{\mu * \gamma_{t,\sigma}}(\lambda)||_{HS} \leq \lim_{|\lambda|\to\infty} |\lambda|^p d_\lambda e^{-t\sigma\kappa_\lambda} \leq C \lim_{|\lambda|\to\infty} |\lambda|^{p+m} e^{-t\sigma|\lambda|^2} = 0,$$

and the result follows from Theorem 4.5.3. □

4.6 Idempotents and Convolution Powers

We say that $\mu \in \mathcal{P}(G)$ is *idempotent* if $\mu * \mu = \mu$. Equivalently by Theorems 4.2.1 and 4.2.3, μ is idempotent if and only if $\widehat{\mu}(\pi)^2 = \widehat{\mu}(\pi)$ for all $\pi \in \widehat{G}$. It is easy to see that normalised Haar measure m on G is idempotent. More generally let H be a closed subgroup of G and let $m_H^{(0)}$ denote its normalised Haar measure. We extend $m_H^{(0)}$ to a measure $m_H \in \mathcal{P}(G)$ that has support H by the prescription

$$m_H(B) = m_H^{(0)}(B \cap H)$$

for all $B \in \mathcal{B}(G)$. For example if $H = \{e\}$ then $m_H = \delta_e$. It is again easy to see that m_H is always idempotent. The following result is due to Wendel:

Theorem 4.6.1. *If $\mu \in \mathcal{P}(G)$ is idempotent then $\mu = m_H$ for some closed subgroup H of G. Moreover $H = supp(\mu)$.*

Proof. Let $H := \text{supp}(\mu)$ then by (4.1.2) we have $H = H^2$ and so H is a semigroup under the group law. It is also closed and hence compact. It is known that any subset of G that has these properties is a subgroup (see e.g. Lemma 2 in Gelbaum et. al. and also Corollary 1.2.9. on p.34 of Heyer.) Now let $f \in C(H)$ and define for each $h \in H$:

$$A_f(h) = \int_G f(gh)\mu(dg).$$

Using the uniform continuity of left translation it is easily verified that $A_f \in C(H)$. Now let h_0 be the point in G where A_f attains its maximum value. Then

$R_{h_o}f$ attains its maximum at e. From now on we denote $f_1 := R_{h_o}f$. Then since μ is idempotent:

$$\begin{aligned} A_{f_1}(e) &= \int_H f_1(g)\mu(dg) \\ &= \int_H f_1(g)(\mu * \mu)(dg) \\ &= \int_H \int_H f_1(g_1 g_2)\mu(dg_1)\mu(dg_2) \\ &= \int_H A_{f_1}(g_2)\mu(dg_2) \\ &\leq A_{f_1}(e). \end{aligned}$$

Hence we see that $\int_G (A_{f_1}(e) - A_{f_1}(g_2)\mu(dg_2) = 0$ and so $A_{f_1}(e) - A_{f_1}(g)$ for all $g \in H$. It follows by uniqueness of Haar measure and the fact that $\mu(H) = 1$ that $\mu = m_H$ as required. □

Let $\mu \in \mathcal{P}(G)$ and $n \in \mathbb{N}$. We define the *nth convolution power* of μ to be $\mu^{*(n)} = \mu * \cdots \mu$ (n times.) Note that we than have for all $\pi \in \widehat{G}, \widehat{\mu^{*(n)}}(\pi) = \widehat{\mu}(\pi)^n$. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(X_n, n \in \mathbb{N})$ be a sequence of independent, identically distributed (or i.i.d.) G - valued random variables. Let $(S_n, n \in \mathbb{N})$ be the associated G-valued *random walk* so that for each $n \in \mathbb{N}, S_n = X_1 X_2 \ldots X_n$. Then the law of S_n is precisely $\mu^{*(n)}$. It is of interest to study the asymptotic behaviour of the random walk as for large n. In particular we might consider the weak limit of $\mu^{*(n)}$ as $n \to \infty$. It is clear that if μ is regular and the limit exists it is an idempotent and so by Theorem 4.6.1

$$\lim_{n \to \infty} \mu^{*(n)} = m_H,$$

for some closed subgroup H of G.

Necessary and sufficient conditions for the limit to exist were found by Stromberg. We quote his result but omit the proof.

Theorem 4.6.2. *Let $\mu \in \mathcal{P}(G)$ be regular and let K be the smallest closed subgroup of G containing supp(μ). Then $\lim_{n\to\infty} \mu^{*(n)}$ exists if and only if supp(μ) is not contained in any proper closed subgroup of K.*

Kawada and Itô established an *equidistribution theorem* which gives conditions for the limit to exist and be normalised Haar measure m on the whole group. First we need a definition. We say that $\mu \in \mathcal{P}(G)$ is *aperiodic* if supp(μ) is not contained in a left or right coset of a proper closed subgroup of G. We then have the following:

Theorem 4.6.3 (Kawada-Itô equidistribution theorem)**.** *If $\mu \in \mathcal{P}(G)$ is aperiodic then $(\mu^{*(n)}, n \in \mathbb{N})$ converges weakly to normalised Haar measure.*

Proof. By Theorem 4.2.6, it is sufficient to show that $\lim_{n\to\infty} \widehat{\mu}(\pi)^n = 0$ for all non-trivial $\pi \in \widehat{G}$. This is clearly equivalent to the requirement that all the eigenvalues of $\widehat{\mu}$ have modulus strictly less that 1. Note that since $\widehat{\mu}(\pi)$ is a contraction, it's eigenvalues cannot exceed 1. Now let λ be an eigenvalue of $\widehat{\mu}(\pi)$. Then we can find a unitary matrix U_π acting in V_π so that

$$U_\pi \widehat{\mu}(\pi) U_\pi^{-1} = \begin{pmatrix} \lambda & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & & & & & \\ \cdot & & & & & \\ \cdot & & & D_\pi & & \\ \cdot & & & & & \\ 0 & & & & & \end{pmatrix},$$

where D_π is some $(d_\pi - 1) \times (d_\pi - 1)$ matrix. In particular, we have

$$\lambda = (U_\pi \widehat{\mu}(\pi) U_\pi^{-1})_{11} = \int_G (U_\pi \pi(g^{-1}) U_\pi^{-1})_{11} \mu(dg).$$

Now suppose that $|\lambda| = 1$. Then we must have $(U_\pi \pi(g^{-1}) U_\pi^{-1})_{11} = \lambda$ for all $g \in G$ for which $g^{-1} \in \text{supp}(\mu)$. For suppose that this not the case. Then there exists an open set E in $\text{supp}(\mu)$ for which $0 < \left| \int_G (U_\pi \pi(g^{-1}) U_\pi^{-1})_{11} \mu(dg) \right| < 1$. But then

$$\begin{aligned} \lambda &= \int_{E^c} (U_\pi \pi(g^{-1}) U_\pi^{-1})_{11} \mu(dg) + \int_E (U_\pi \pi(g^{-1}) U_\pi^{-1})_{11} \mu(dg) \\ &= \lambda + \int_E (U_\pi \pi(g^{-1}) U_\pi^{-1})_{11} \mu(dg), \end{aligned}$$

and this yields the desired contradiction. Now suppose that $\lambda = 1$. Then we have

$$\text{supp}(\mu) \subseteq H := \left\{ g \in G, U\pi(g^{-1})U = \begin{pmatrix} \lambda & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & & & & & \\ \cdot & & & & & \\ \cdot & & & E_\pi & & \\ \cdot & & & & & \\ 0 & & & & & \end{pmatrix} \right\},$$

where E_π is some $(d_\pi - 1) \times (d_\pi - 1)$ matrix. But H is a proper closed subgroup of G and this contradicts aperiodicity of μ.

Now suppose that $\lambda = e^{i\theta}$ for some $\theta \in \mathbb{R} - 2\pi\mathbb{Z}$. Then arguing as above we can find a unitary matrix V_π so that $e^{i\theta} = (V_\pi \widehat{\mu}(\pi) V_\pi^{-1})_{11} = \int_G (V_\pi \pi(g^{-1}) V_\pi^{-1})_{11} \mu(dg)$. and

$$\text{supp}(\mu) \subseteq \Gamma := \left\{ g \in G, U\pi(g^{-1})U = \begin{pmatrix} e^{i\theta} & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & & & & & \\ \cdot & & & & & \\ \cdot & & & F_\pi & & \\ \cdot & & & & & \\ 0 & & & & & \end{pmatrix} \right\},$$

where F_π is some $(d_\pi - 1) \times (d_\pi - 1)$ matrix. Now let $g_0 \in G$ be such that $(V_\pi \pi(g_0^{-1}) V_\pi^{-1})_{11} = e^{i\theta}$. Then it is easily verified that $\Gamma = g_0 H$ and this again yields a contradiction with aperiodicity. The required result follows. □

4.7 Convolution Operators

in this and the next section, we will drop the condition that G be a compact Lie group. We work more generally and assume throughout that G is a locally compact, separable and second countable topological group. Convolution operators were already introduced in section 4.4 for compact Lie groups. Now we study them more systematically. Let $\mu \in \mathcal{P}(G)$. The associated *right convolution operator* $P_\mu^{(R)}$ is defined on $B_b(F)$ by the prescription $P_\mu^{(R)} f = f * \mu$ for $f \in B_b(G)$ so that

$$(P_\mu^{(R)} f)(g) = \int_G f(gh)\mu(dh),$$

for all $g \in G$. Similarly the left convolution operator $P_\mu^{(L)}$ is defined by

$$(P_\mu^{(L)} f)(g) = \int_G f(hg)\mu(dh).$$

The reader may check that these are related by the identity

$$P_\mu^{(L)} f = \widetilde{P_{\widetilde{\mu}}^{(R)} \widetilde{f}}.$$

Furthermore if $\mu_1, \mu_2 \in \mathcal{P}(G)$ then $P_{\mu_1}^{(R)} P_{\mu_2}^{(L)} = P_{\mu_2}^{(L)} P_{\mu_1}^{(R)}$. We also have that for all $\mu \in \mathcal{P}(G), g \in G, L_g P_\mu^{(R)} = P_\mu^{(R)} L_g$ and $R_g P_\mu^{(L)} = P_\mu^{(L)} R_g$.

From now on we will almost always work with $P_\mu^{(R)}$ which we will denote simply as P_μ.

Proposition 4.7.1. *For each $\mu \in \mathcal{P}(G)$,*

1. *P_μ is linear,*
2. *$P_\mu 1 = 1$,*
3. *If $f \geq 0$ then $P_\mu f \geq 0$,*
4. *P_μ is a contraction,*
5. *$P_\mu : C_0(G) \to C_0(G)$,*
6. *If $\mu_1, \mu_2 \in \mathcal{P}(G)$ then*

$$P_{\mu_1 * \mu_2} = P_{\mu_1} P_{\mu_2}.$$

Proof. (1) to (4) are all easy exercises. For (5) first note that continuity is straightforward to establish by using a dominated convergence argument (or see the discussion before Theorem 4.4.2.) To see that if f vanishes at infinity then so does $P_\mu f$, let $G_\infty := G \cup \{\infty\}$ be the one point compactification of G. Then $f \in C_0(G)$ if and only if $\lim_{g \to \infty} f(g) = 0$. Then since $R_h : C_0(G) \to C_0(G)$ for all $h \in G$, the result follows by using the dominated convergence theorem. To show (6), let $f \in B_b(G)$ and $g \in G$ then by Fubini's theorem,

$$\begin{aligned}
(P_{\mu_1 * \mu_2} f)(g) &= \int_G f(gh)(\mu_1 * \mu_2)(dh) \\
&= \int_G \int_G f(gh_1h_2)\mu_1(dh_1)\mu_2(dh_2) \\
&= \int_G \left(\int_G f(gh_1h_2)\mu_2(dh_2) \right) \mu_1(dh_1) \\
&= \int_G (P_{\mu_2} f)(gh_1)\mu_1(dh_1) \\
&= (P_{\mu_1} P_{\mu_2} f)(g).
\end{aligned}$$

□

Note that if G is compact then we use $C(G)$ in place of $C_0(G)$ in Proposition 4.7.1 (5).

There is a useful link between the convolution operator and Fourier transform of the measure on compact groups.

Proposition 4.7.2. *If μ is a Borel probability measure on a compact Lie group G, then for all $\pi \in \widehat{G}, 1 \leq i, j \leq d_\pi$*

$$\widehat{\widetilde{\mu}}(\pi)_{ij} = P_\mu \pi_{ij}(e).$$

Proof.

$$\begin{aligned}
P_\mu \pi_{ij}(e) &= \int_G \pi_{ij}(\tau)\mu(d\tau) \\
&= \int_G \pi_{ij}(\tau^{-1})\widetilde{\mu}(d\tau) \\
&= \widehat{\widetilde{\mu}}(\pi)_{ij} \qquad \square
\end{aligned}$$

Returning to the general case, we easily deduce from Proposition 4.7.1 (6) that for all $\mu \in \mathcal{P}(G), n \in \mathbb{N}$,

$$P_{\mu^{*(n)}} = P_\mu^n. \tag{4.7.11}$$

Note also that P_μ^n is the n-step transition operator for the random walk $(S(n), n \in \mathbb{N})$, indeed for all $f \in B_b(G), g \in G$,

$$(P_\mu^n f)(g) = \mathbb{E}(f(gS(n))).$$

In the following we fix a right Haar measure m_R on G and consider the p-norms $||\cdot||_p$ in the spaces $L^p(G, m_R)$ for $1 \leq p < \infty$.

Proposition 4.7.3. *For all $\mu \in \mathcal{P}(G), f \in C_c(G), 1 \leq p < \infty$,*

$$||P_\mu f||_p \leq ||f||_p.$$

Proof. By Jensen's inequality and Fubini's theorem

$$\begin{aligned}
||P_\mu f||_p^p &= \int_G |P_\mu f(g)|^p m_R(dg) \\
&= \int_G \left|\int_G f(gh)\mu(dh)\right|^p m_R(dg) \\
&\leq \int_G \int_G |f(gh)|^p \mu(dh) m_R(dg) \\
&= \int_G \int_G |f(gh)|^p m_R(dg)\mu(dh) \\
&= \int_G |f(g)|^p m_R(dg) = ||f||_p^p.
\end{aligned}$$

□

We have just shown that P_μ is a contraction from $C_c(G)$ to $L^p(G, m_R)$ and so it extends to a contraction on the whole of $L^p(G, m_R)$. We will continue to denote the operator by the same symbol P_μ whenever we consider it as acting in the L^p-spaces. For the case $p = 2$ the reader may verify the useful result

$$P_\mu^* = P_{\widetilde{\mu}}. \tag{4.7.12}$$

Let f be a non-negative measurable function defined on G. It is said to be *μ-harmonic* if $P_\mu f = f$, *μ-superharmonic* if $P_\mu f \leq f$ and *μ-subharmonic* if $P_\mu f \geq f$. Clearly non-negative constant functions are μ-harmonic. In the next section, we will investigate a condition under which they are the only ones. Note that if $(S_n, n \in \mathbb{N})$ is the random walk associated to μ and f is bounded and μ-superharmonic then $(f(S_n), n \in \mathbb{N})$ is a supermartingale with respect to the filtration $(\mathcal{F}_n, n \in \mathbb{N})$ of $\mathcal{F}$ where for each $n \in \mathbb{N}, \mathcal{F}_n := \sigma\{X_1, \ldots, X_n\}$. To see this observe that by the Markov property.

$$\mathbb{E}(f(S_{n+1})|\mathcal{F}_n) = (P_\mu f)(S_n) \leq f(S_n).$$

Similarly we obtain a martingale (respectively, submartingale) if f is μ-harmonic (respectively, μ-subharmonic.)

Now let $\mathcal{M}(G)$ be the space of all Radon measures on G. For each $\mu \in \mathcal{P}(G)$, we may consider the dual action P_μ^* of P_μ on $\mathcal{M}(G)$ which is defined by the prescription:

$$(P_\mu^* \rho)(f) = \rho(P_\mu f),$$

for all $\rho \in \mathcal{M}(G), f \in C_c(G)$. We generalise the definitions we gave above for functions and say that $\rho \in \mathcal{M}(G)$ is μ-harmonic if $P_\mu^* \rho = \rho$, μ-superharmonic if $P_\mu^* \rho \leq \rho$ and μ-subharmonic if $P_\mu^* \rho \geq \rho$.[4] Now let $\rho \in \mathcal{M}(G)$ be absolutely continuous with respect to m_R with $f_\rho := \dfrac{d\rho}{dm_R}$. It is left as an exercise for the reader to check that f_ρ is μ-superharmonic if and only if ρ is $\widetilde{\mu}$-superharmonic.

We give some useful properties of μ-superharmonic functions.

[4] If $\rho_1, \rho_2 \in \mathcal{M}(G)$ we write $\mu_1 \geq \mu_2$ if $\mu_1(f) \geq \mu_2(f)$ for all $f \in C_c(G)_+$.

Proposition 4.7.4. *Let $\mu \in \mathcal{P}(G)$*

1. *Suppose that $\lambda, \rho \in \mathcal{M}(G)$ wherein λ has compact support. If ρ is μ-superharmonic then so is $\lambda * \rho$.*

2. *If f is μ-superharmonic then so is $f \wedge c$ for any $c > 0$.*

Proof.

1. For all $f \in C_c(G)_+$ we have

$$\begin{aligned} P_\mu^*(\lambda * \rho)(f) &= \int_G \int_G (P_\mu^{(R)} f)(gh)\lambda(dg)\rho(dh) \\ &= \int_G (P_\lambda^{(L)} P_\mu^{(R)} f)(h)\rho(dh) \\ &= \int_G (P_\mu^{(R)} P_\lambda^{(L)} f)(h)\rho(dh) \\ &= (P_\mu^* \rho)(P_\lambda^{(L)} f) \\ &\leq \rho(P_\lambda^{(L)} f) = (\lambda * \rho)(f). \end{aligned}$$

2. This follows from the easily verified fact that $P_\mu(f \wedge c) \leq P_\mu f \wedge c$. □

Next we establish a connection between properties of convolution operators and existence and regularity of densities as discussed in section 4.5. This result will be useful for us in the next chapter. Readers requiring background on Hilbert-Schmidt operators are referred to Appendix 3.

Theorem 4.7.1. *Let $\mu \in \mathcal{P}(G)$. The operator P_μ acting in $L^2(G)$ is Hilbert-Schmidt if and only if μ has a square-integrable density.*

Proof. For sufficiency assume that μ has density $f_\mu \in L^2(G)$. Then for all $g \in L^2(G), \sigma \in G, (P_\mu g)(\sigma) = \int_G g(\sigma\tau) f_\mu(\tau) d\tau = \int_G g(\tau) f_\mu(\sigma^{-1}\tau) d\tau$. Now define the mapping $k_\mu : G \times G \to \mathbb{R}$ by $k_\mu(\sigma, \tau) := f_\mu(\sigma^{-1}\tau)$. Then $k_\mu \in L^2(G \times G)$ and the result follows by Theorem 4.7.1 in Appendix 3. For necessity, suppose that P_μ is Hilbert-Schmidt. Then it has a kernel $k \in L^2(G \times G)$ and

$$(P_\mu f)(\sigma) = \int_G f(\tau) k_\mu(\sigma, \tau) d\tau.$$

In particular for each $A \in \mathcal{B}(G)$,

$$\mu(A) = P_\mu 1_A(e) = \int_A k_\mu(e, \tau) d\tau.$$

Then for all $g \in C(G, \mathbb{R}), \int_G g(\sigma)\mu(d\sigma) = \int_G g(\sigma) k_\mu(e, \sigma) d\sigma$. It then follows by the argument used in the last part of the proof of Theorem 4.5.1 that μ is absolutely continuous with respect to m with density $f_\mu := k_\mu(e, \cdot)$ and we also have $f_\mu \in L^2(G)$. □

A linear operator $T : B_b(G) \to B_b(G)$ is said to be *strong Feller* if $\mathrm{Ran}(T) \subseteq C_b(G)$. The result of the next theorem is obtained by Hawkes for $G = \mathbb{R}^n$ where a more detailed analysis appears to be possible.

Theorem 4.7.2. *If $\mu \in \mathcal{P}(G)$ has a continuous density g_μ with respect to left Haar measure then the convolution operator P_μ is strong Feller.*

Proof. We need only establish continuity. Let $\sigma_1, \sigma_2 \in G$. Then for all $f \in B_b(G)$,

$$\begin{aligned}
|P_\mu f(\sigma_1) - P_\mu f(\sigma_2)| &= \left| \int_G f(\sigma_1 \tau)\mu(d\tau) - \int_G f(\sigma_2 \tau)\mu(d\tau) \right| \\
&\leq \int_G |f(\tau)||g_\mu(\sigma_1^{-1}\tau) - g_\mu(\sigma_2^{-1}\tau)|d\tau \\
&\leq \sup_{\tau \in G} |f(\tau)| \int_G |g_\mu(\sigma_1^{-1}\tau) - g_\mu(\sigma_2^{-1}\tau)|d\tau
\end{aligned}$$

Let $A_{\sigma_1,\sigma_2} := \{\tau \in G; g_\mu(\sigma_1^{-1}\tau) \geq g_\mu(\sigma_2^{-1}\tau)\}$. Then $A_{\sigma_1,\sigma_2} \in \mathcal{B}(G)$ and for all $\sigma_1, \sigma_2, \tau \in G$

$$\begin{aligned}
|g_\mu(\sigma_1^{-1}\tau) - g_\mu(\sigma_2^{-1}\tau)| &= (g_\mu(\sigma_1^{-1}\tau) - g_\mu(\sigma_2^{-1}\tau))1_{A_{\sigma_1,\sigma_2}}(\tau) \\
&+ (g_\mu(\sigma_2^{-1}\tau) - g_\mu(\sigma_1^{-1}\tau))1_{A^c_{\sigma_1,\sigma_2}}(\tau) \\
&\leq g_\mu(\sigma_1^{-1}\tau)1_{A_{\sigma_1,\sigma_2}}(\tau) + g_\mu(\sigma_2^{-1}\tau)1_{A^c_{\sigma_1,\sigma_2}}(\tau) \\
&\leq g_\mu(\sigma_1^{-1}\tau) + g_\mu(\sigma_2^{-1}\tau).
\end{aligned}$$

Now let $(\sigma_n, n \in \mathbb{N})$ be a sequence in G which converges to σ_1 as $n \to \infty$. Then given an open set U containing σ_1 in G that has compact closure we can find $N \in \mathbb{N}$ so that $\sigma_n \in U$ for all $n > N$. Define $h_{g,N}(\tau) := g_\mu(\sigma_1^{-1}\tau) + \sup_{\rho \in \overline{U}} g_\mu(\rho^{-1}\tau)$ for $\tau \in G$. Then $h_{g,N}$ is clearly measurable and $\int_G h_{g,N}(\tau)d\tau = 2\int_G g_\mu(\tau)d\tau = 2$. From the estimates we computed earlier we see that for all $n > N, \tau \in G, |g_\mu(\sigma_1^{-1}\tau) - g_\mu(\sigma_n^{-1}\tau)| \leq h_{g,N}(\tau)$. We may then use dominated convergence to prove the desired continuity. □

Finally we establish a useful spectral property for the case where μ is a central measure and G is compact.

Theorem 4.7.3. *If G is compact and μ is a central probability measure, then $\{\pi_{ij}, 1 \leq i, j \leq d_\pi, \pi \in \widehat{G}\}$ are a complete set of eigenfunctions for P_μ acting in $L^2(G)$. Moreover we have*

$$P_\mu \pi_{ij} = \overline{c_\pi} \pi_{ij},$$

for all $1 \leq i, j \leq d_\pi, \pi \in \widehat{G}$ where $\widehat{\mu}(\pi) = c_\pi I_\pi$.

Proof. For all $\sigma \in G$ we have

$$\begin{aligned}
P_\mu \pi_{ij}(\sigma) &= \int_G \pi_{ij}(\sigma\tau)\mu(d\tau) \\
&= \sum_{k=1}^{d_\pi} \pi_{ik}(\sigma) \int_G \pi_{kj}(\tau)\mu(d\tau) \\
&= \sum_{k=1}^{d_\pi} \pi_{ik}(\sigma)\widehat{\widetilde{\mu}(\pi)}_{kj} \\
&= \sum_{k=1}^{d_\pi} \pi_{ik}(\sigma)\widehat{\mu(\pi)}^*_{kj} \\
&= \sum_{k=1}^{d_\pi} \pi_{ik}\overline{c_\pi}\delta_{kj} \\
&= \overline{c_\pi}\pi_{ij}(\sigma),
\end{aligned}$$

and the result follows. □

4.8 Recurrence

If $\mu \in \mathcal{P}(G)$ we define $\mu^{*(0)} = \delta_e$. Throughout this section we will assume that $\mu \in \mathcal{P}(G)$ is *full*, i.e. the smallest closed subgroup of G that is generated by $\mathrm{supp}(\mu)$ is G. We define the *potential measure* V_μ of μ (when it exists) by the prescription

$$V_\mu(f) := \sum_{n=0}^{\infty} \mu^{*(n)}(f),$$

for each $f \in C_c(G)$ so that $V_\mu(f) = \sum_{n=0}^{\infty} \mu(P_\mu^n f)$.

We say that μ is *transient* if $V_\mu(A) < \infty$ for all open relatively compact subsets of G and *recurrent* if $V_\mu(A) = \infty$ for all non-empty open subsets of G. We say that the group G is *recurrent* if $\mathcal{P}(G)$ contains at least one full recurrent measure.

Theorem 4.8.1. *[Recurrence-Transience Dichotomy] Every full $\mu \in \mathcal{P}(G)$ is either recurrent or transient.*

We omit the proof which can be obtained by combining the results of Theorem 22 (pp.19-20) and Theorem 26 (pp. 23-4) in Guivarc'h et al.

From the random walk point of view, recurrence is equivalent to the requirement that for all $g \in G, P\left(\lim_{n\to\infty} S_n \in V_g\right) = 1$ for any neighbourhood V_g of g. From the point of view of this monograph a key result is the following

Proposition 4.8.1. *Every compact group G is recurrent.*

Proof. By the recurrence-transience dichotomy (Theorem 4.8.1), if there exists a full $\mu \in \mathcal{P}(G)$ with $\mu(A) = \infty$ for some open relatively compact subset A of G then G cannot be transient and so must be recurrent. But we may take $A = G$ and then for any $\mu \in \mathcal{P}(G), V_\mu(G) = \sum_{n=0}^{\infty} \mu^{*(n)}(G) = \infty$. □

Note that the last result shows that every random walk on a compact G whose step-size has full measure is recurrent.

Lemma 4.8.1. *Let μ be a full measure in $\mathcal{P}(G)$ and f be μ-superharmonic. If μ is recurrent then f is μ-harmonic.*

Proof. We suppose that $f - P_\mu f > 0$ and seek a contradiction. By the recurrence assumption

$$\infty = V_\mu(f - P_\mu f) = \sum_{n=0}^{\infty} \mu(P_\mu^n(f - P_\mu f),$$

and so

$$\infty = \sum_{n=0}^{\infty} P_\mu^n(f - P_\mu f) = \lim_{n\to\infty}(f - P_\mu^n f) \leq f,$$

and this yields the required contradiction. □

The main result of this section is the following. Our proof closely mirrors that of Guivarc'h et al Proposition 45, pp.42-4.

Theorem 4.8.2. *Let $\mu \in \mathcal{P}(G)$ be full. The following are equivalent.*

(i) μ is recurrent.

(ii) Every μ-superharmonic continuous function on G is constant.

(iii) Every μ-superharmonic measure is a right Haar measure.

(iv) Every μ-superharmonic function on G is constant m_R-almost everywhere.

Proof. (i) ⇒ (ii). Let f be a μ-superharmonic function on G. By Lemma 4.8.1 it is μ-harmonic. Assume that f is bounded. By continuity there exists $g_0 \in G$ so that $f(g_0) = \sup_{g\in G} f(g)$. Then we have

$$f(g_0) = (P_\mu f)(g_0) = \int_G f(g_0 h)\mu(dh),$$

and so

$$\int_G (f(g_0) - f(g_0 h))\mu(dh) = 0.$$

It follows that $f(g_0) = f(g_0 h)$ for all $h \in G$ and hence f is constant. To extend this result to the non-bounded case, observe that we can replace f by $f \wedge c$ where $c > 0$ and appeal to Proposition 4.7.4(2).

(ii) $\Rightarrow$ (i). We assume that μ is transitive and seek a contradiction. Let $f \in C_c(G)_+$ be non-trivial and define $F_f = f * V_\mu$ so that for all $g \in G, F_f(g) = \sum_{n=0}^{\infty} f(gh)\mu^{*(n)}(dh)$. By a standard use of dominated convergence, we can see that F_f is continuous. Then it is easily verified that

$$P_\mu F_f = F_f - f \leq F_f,$$

and so F_f is both continuous and μ-superharmonic. Hence it is constant and so $f = P_\mu F_f - F_f = 0$ and that yields the desired contradiction.

(ii) $\Rightarrow$ (iii). Assume that $\rho \in \mathcal{M}(G)$ is such that $\widetilde{\rho}$ is μ-superharmonic. Choose $\phi \in C_c(G)_+$ and define $G_{\phi,\rho}(x) := \int_G \phi(xy)\widetilde{\rho}(dy)$ for all $x \in G$. By Fubini's theorem,

$$\begin{aligned} P_\mu G_{\phi,\rho}(x) &= \int_G \int_G \phi(xhy)\mu(dh)\widetilde{\rho}(dy) \\ &= (P_\mu^* \widetilde{\rho})(L_{x^{-1}}\phi) \\ &\leq \widetilde{\rho}(L_{x^{-1}}\phi) = G_{\phi,\rho}(x) \end{aligned}$$

for all $x \in G$. So $G_{\phi,\rho}$ is superharmonic and hence constant. So $\int_G \phi(xy)\widetilde{\rho}(dy) = \int_G \phi(y)\widetilde{\rho}(dy)$ for all $x \in G$ and all $\phi \in C_c(G)_+$. It follows that $\widetilde{\rho}$ is a left Haar measure and so ρ is a right Haar measure.

(iii) $\Rightarrow$ (iv). Let $f \in B_b(G)_+$ and consider the measure $\rho_f \in \mathcal{M}(G)$ for which $\dfrac{d\rho_f}{dm_R} = f$. If f is μ-superharmonic then ρ_f is $\widetilde{\mu}$-superharmonic and so ρ_f is a right Haar measure. It follows that f is constant almost everywhere with respect to m_R. If f is not bounded, we can again replace f by $f \wedge c$ where $c > 0$ and use Proposition 4.7.4(2).

(iv) $\Rightarrow$ (ii) is obvious. □

Let M be a topological space and let $\mathcal{P}(M)$ be the space of all Borel probability measures on M. A continuous mapping $\alpha : G \times M \to M$ for which $\alpha(e, x) = x$ for all $x \in X$ is called an *action* of G on M. An action is *transitive* if for all $g_1, g_2 \in G, x \in M, \alpha(g_1, \alpha(g_2, x)) = \alpha(g_1 g_2, x)$. A group G is said to be *amenable in action* if for every transitive action α on every compact space M, there exists a Borel probability measure μ_M on M such that for all $f \in C(M), g \in G$

$$\int_M f(\alpha(g, x))\mu_M(dx) = \int_G f(x)\mu_M(dx).$$

The proof of the following theorem is based on that in Guivarc'h et al (pp 45-7). We will need the notation of the convolution of $\mu \in \mathcal{P}(G)$ and $\lambda \in \mathcal{P}(G)$ relative to a given transitive action α. This is precisely the measure $\mu *_\alpha \lambda \in \mathcal{P}(M)$ such that for all $A \in \mathcal{B}(M)$:

$$(\mu *_\alpha \lambda)(A) := \int_M \int_G 1_A(\alpha(g, x))\mu(dg)\lambda(dx).$$

Theorem 4.8.3. *Every recurrent group is amenable in action.*

Proof. Let μ be a full recurrent probability measure on G and $\nu \in \mathcal{P}(M)$ be arbitrary. For each $n \in \mathbb{N}$ define $\mu_n := \frac{1}{n}\sum_{k=0}^{n-1} \mu^{*(k)}$. Then $(\mu_n *_\alpha \nu, n \in \mathbb{N})$ is weakly relatively compact and so has a subsequence $(\mu_{n_k} *_\alpha \nu, k \in \mathbb{N})$ that converges weakly to some $\lambda \in \mathcal{P}(M)$. If we can prove that λ is invariant then we are done. We first show that $\mu *_\alpha \lambda = \lambda$. Indeed we have for all $f \in C(X)$, using the transitivity of α,

$$
\begin{aligned}
(\mu *_\alpha \lambda)(f) &= \int_M \int_G f(\alpha(g,x))\mu(dg)\lambda(dx) \\
&= \lim_{k\to\infty} \int_M \int_G \int_G f(\alpha(gh,x))\mu(dg)\mu_k(dh)\nu(dx) \\
&= \lim_{k\to\infty} \int_M f(x)[(\mu * \mu_k) *_\alpha \nu](dx) \\
&= \lim_{k\to\infty} \int_M f(x)(\mu_k) *_\alpha \nu)(dx) = \lambda(f).
\end{aligned}
$$

To see that λ is indeed invariant, let $f \in C(X)_+$ and define $\Phi_f \in C(G)_+$ by $\Phi_f(g) = \int_X f(\alpha(g,x))\lambda(dx)$ for $g \in G$. Then Φ_f is μ-harmonic, for by Fubini's theorem

$$
\begin{aligned}
P_\mu \Phi_f(g) &= \int_M \int_G f(\alpha(gh,x))\mu(dh)\lambda(dx) \\
&= \int_M f(\alpha(g,x))(\mu *_\alpha \lambda)(x) \\
&= \int_M f(\alpha(g,x))\lambda(dx) = \Phi_f(g).
\end{aligned}
$$

So by Theorem 4.8.2, Φ_f is constant and hence for all $g \in G$,

$$
\int_M f(\alpha(g,x))\lambda(dx) = \Phi_f(g) = \Phi_f(e) = \int_M f(\alpha(e,x))\lambda(dx) = \int_M f(x)\lambda(dx),
$$

and the result follows. ☐

For example, let H be a closed subgroup of a compact group G and $M = G/H$ be the (compact) homogeneous space of left cosets. Define the *natural action* of G on M by $\alpha(g, g'H) = gg'H$ for all $g, g' \in G$. This is clearly continuous and transitive. G is recurrent by Proposition 4.8.1 and so by Theorem 4.8.3 we can assert the existence of $\mu_M \in \mathcal{P}(M)$ so that for all $g \in G$

$$
\int_M f(gx)\mu_M(dx) = \int_M f(x)\mu_M(dx).
$$

For example, take $G = SO(n), H = SO(n-1)$ so that $M = S^{n-1}$. In this case μ_M is the normalised surface measure σ_{n-1} for which we have the recursive formula

$$\int_{S^{n-1}} f(x)\sigma_{n-1}(dx)$$
$$= \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)} \int_0^\pi \left(\int_{S^{n-2}} f(\sin(\theta y) + \cos(\theta)e_n)\sigma_{n-2}(dy)\right) \sin^{n-2}(\theta)d\theta,$$

for all $f \in C(S^{n-1})$ where e_n is the "north pole" in S^{n-1} (see Faraut pp.186-90 for details.)

4.9 Notes and Further Reading

The interaction between probability and group theory covers a huge area and it is difficult to do this justice in such a short space. In particular this includes probability on discrete groups (which is not really the topic of this monograph) and has considerable overlap with "stochastic differential geometry" as random motion on a Lie group can be regarded as a special case of that on a more general manifold.

In the 1960s a considerable literature has begun to evolve on probability theory in such general mathematical structures as groups on the one hand and Banach spaces on the other. These two directions have now diverged considerably but in 1963 Grenander was able to justify including both themes within a single volume. From the continuous group point of view, he introduced the Fourier transform and gave some attention to limit theorems. He traces the historic roots of the subject back to work by Perrin in 1928 on Brownian motion in the rotation group. Hannan's survey paper from 1965 is also of historical interest. He develops applications to second order stationary processes, experimental design and ANOVA. Parthasarathy's book on 1967 discusses probability on metric groups and gives an account of infinite divisibility, the Lévy-Khintchine formula and the central limit theorem on locally compact abelian groups. Ten years later, Heyer's highly influential treatise appeared which gave a comprehensive and detailed state of the art account of probability on (general) locally compact groups. This monograph is now a classic and after 35 years is still a highly valuable resource for those doing research in this area. Highlights are the treatment of Hunt's classification of the generators of convolution semigroups and the central limit theorem. We will investigate both of these topics in the next chapter. Ten years later, Diaconis published his beautiful lecture notes that demonstrate the fruitfulness of group actions in a variety of contexts with probability and statistics from card shuffling to ANOVA and spectral analysis of time series.

In more recent years there have been a number of books and monographs on more specific topics concerned with probability on groups. For example Hazod and Siebert study stable laws on locally compact groups (where these make sense), Neuenschwander investigates limit theorems and Brownian motion on the Heisenberg group and Liao's monograph is devoted to Lie group-valued Lévy processes.