A Wavelet Construction for Quantum Brownian Motion and Quantum Brownian Bridges

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Dedicated to Robin Hudson on his 65th birthday.

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1. Probabilistic and Stochastic Hilbertian Structures
2. Wavelet Construction of Quantum Brownian Motion.
3. The Quantum Brownian Bridge

Two key references:

A.M.Cockroft, R.L.Hudson, Quantum mechanical Wiener processes, *J. Mult. Anal.* **7**, 107-24 (1977)

C.D.Cushen, R.L.Hudson, A quantum-mechanical central limit theorem, *J. Appl. Prob.* **8**, 454-69 (1971)

1 Probabilistic and Stochastic Hilbertian Structures

Motivation: A *quantum random variable* in the sense of Accardi, Lewis and Frigerio (1982) is a $*$-homomorphism j between involutive algebras $\mathcal{A}$ and $\mathcal{B}$, where B is equipped with a state to determine expectations.

A "quantum random variable" in the sense of Cockcroft and Hudson (1977) is (implicitly) a pair of self-adjoint operators acting in a Hilbert space.

Axiomatising the Cockcroft/Hudson approach gives a coherent formalism for *quantum Brownian motions/bridges.*

1.1 Daggered Spaces

Let H be a complex separable Hilbert space and $\mathcal{D}$ be a dense linear subspace in H. We will be interested in linear operators T defined on H which have the following properties.

(i) $\mathcal{D} \subseteq \mathrm{Dom}(T)$ and the restriction of T to $\mathcal{D}$ is closable.

(ii) $\mathcal{D} \subseteq \mathrm{Dom}(T^*)$.

(iii) $T\mathcal{D} \subseteq \mathcal{D}$.

(iv) $\mathrm{Ran}(T^\dagger) \subseteq \mathrm{Dom}(T)$, where $T^\dagger$ denotes the restriction of T^* to $\mathcal{D}$.

In the sequel, we will often employ the notation $T^\#$ to mean T or $T^\dagger$.

Let $T_1, \cdots, T_n$ be linear operators satisfying (i) to (iv) above.

We denote as $\mathcal{L}_n(\mathcal{D})$ the complex $2n$-dimensional linear space generated by $\{T_1, \cdots, T_n, T_1^\dagger, \ldots, T_n^\dagger\}$.

We call $\mathcal{L}_n(\mathcal{D})$ a *daggered space* of order n.

Slightly abusing terminology, $\{T_1, \ldots, T_n\}$ are called the *generators* of this space.

Such a space is said to be *symmetric* if $\sum_{j=1}^{n}(\alpha_j T_j^{\dagger} - \overline{\alpha_j} T_j)$ is essentially skew-adjoint on $\mathcal{D}$ for all $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$.

We define

$$U(\alpha) := \exp\left\{ \left(\sum_{j=1}^{n} (\alpha_j T_j^{\dagger} - \overline{\alpha_j} T_j) \right)^{c} \right\}$$

to be the associated unitary operator in H, (where ^{c} denotes closure).

1.2 Probabilistic and Stochastic Hilbertian Structures

Throughout this section, we fix a dense linear subspace $\mathcal{D}$ of a complex separable Hilbert space H. A *probabilistic Hilbertian structure of order* n or *PHS(n)* is a pair $(\mathcal{L}_n(\mathcal{D}), \psi_0)$ where

- $\mathcal{L}_n(\mathcal{D})$ is a symmetric daggered space,
- ψ_0 is a fixed unit vector in H.

The *characteristic element* of a PHS(n) is the mapping $\phi : \mathbb{C}^n \to \mathbb{C}$ given by

$$\phi(\alpha) = \langle \psi_0, U(\alpha)\psi_0 \rangle.$$

A PHS(n) is said to be *Gaussian* with if there exists $m \in \mathbb{C}^n$ and an $n \times n$ positive definite symmetric matrix C such that

$$\phi(\alpha) = \exp\left\{ -\frac{1}{2}(\overline{\alpha - m})^T C(\alpha - m) \right\}.$$

If C is a multiple of the identity, we say that the PHS is *i.i.d. Gaussian*. The probabilistic motivation for this is that in this case ϕ is the product of n copies of the characteristic element of a fixed Gaussian PHS(1).

Examples include

- classical multivariate Gaussian vectors,
- annihilation/creation operators associated to n independent quantum harmonic oscillators.

Let $\mathcal{I}$ be an index set and $\{X(t), t \in \mathcal{I}\}$ be a family of linear operators in H.

We call the pair $(\{X(t), t \in \mathcal{I}\}, \psi_0)$ a *stochastic Hilbertian structure* or *SHS* if for each $n \in \mathbb{N}$ and for each $t_1, \ldots, t_n \in \mathcal{I}$, $\{X(t_1), \ldots, X(t_n)\}$ generate a symmetric daggered space with respect to $\mathcal{D}$ of order n. This latter space is denoted $\mathcal{L}_{t_1,\ldots,t_n}(\mathcal{D})$. The collection of all $(\mathcal{L}_{t_1,\ldots,t_n}(\mathcal{D}), \psi_0)$s are called the *finite-dimensional distributions* of the SHS.

A *SHS* is said to be *Gaussian* if if all of its finite-dimensional distributions are Gaussian.

Examples include

- (classical) Gaussian probability spaces (which themselves include classical and abstract Wiener spaces, and white noise spaces),
- quantum Wiener integrals,
- more generally annihilation/creation processes in any boson Fock space.

Fix $\mathcal{I} = \mathbb{R}^+$. Following Cockroft and Hudson, we define a *(standard) quantum Brownian motion* to be a SHS for which

(i) $X(0) = 0$.

(ii) For all $s, t \in \mathbb{R}^+$, on $\mathcal{D}$,

$$[X(s), X(t)] = [X(s)^\dagger, X(t)^\dagger] = 0, [X(s), X(t)^\dagger] = (s \wedge t)I.$$

(iii) For all $n \in \mathbb{N}, T > 0$ and all partitions $\mathcal{P} = \{0 \leq t_0 < \cdots < t_n = T\}$ the operators $\{b_{j,T,\mathcal{P}}, 1 \leq j \leq n\}$ generate an i.i.d Gaussian PHS(n) with covariance I, where

$$b_{j,T,\mathcal{P}} := \frac{1}{\sqrt{t_j - t_{j-1}}}(X(t_j) - X(t_{j-1})).$$

Boson Fock Brownian motions are obtained by taking $H = L^2(\mathbb{R}^+)$, each $X(t) = a(1_{[0,t)})$ where $a(f)$ is the annihilation operator associated to $f \in H$, ψ_0 is the vacuum vector and $\mathcal{D}$ is the linear span of the exponential vectors.

Theorem 1.1 (Cockcroft-Hudson) *Any quantum Brownian motion is equivalent to the boson Fock Brownian motion.*

2 The Wavelet Construction

We construct a quantum Brownian motion with index set $\mathcal{I} = [0,1]$.

The *Haar system* is constructed as follows. The mother wavelet is

$$H(t) := \begin{cases} 1 & \text{if } 0 \le t < \frac{1}{2} \\ -1 & \text{if } \frac{1}{2} \le t \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

The daughter wavelets are constructed by scaling and translation,

$$H_n(t) := 2^{\frac{j}{2}} H(2^j t - k), \quad n = 2^j + k, j \ge 0, 0 \le k < 2^j.$$

If we define $H_0(t) := 1$, then $(H_n, n \in \mathbb{Z}_+)$ is a complete orthonormal basis for $L^2([0,1])$.

The *Schauder system* is defined as follows. The mother wavelet is

$$\begin{aligned} \Delta(t) &:= 2\int_0^t H(u)du \\ &= \begin{cases} 2t & \text{if } 0 \le t < \frac{1}{2} \\ 2(1-t) & \text{if } \frac{1}{2} \le t \le 1 \\ 0 & \text{otherwise.} \end{cases}, \end{aligned}$$

and the daughter wavelets are $\Delta_n(t) := \Delta(2^j t - k)$ for $n = 2^j + k$ as above. If we define $\Delta_0(t) := t$, then $(\Delta_n, n \in \mathbb{Z}_+)$ is a Schauder basis for the Banach space $C_0[0,1]$ of continuous functions on $[0,1]$ which vanish at the origin, equipped with the usual supremum norm. Note in particular that

$$\sup_{n \in \mathbb{Z}_+} \sup_{t \in [0,1]} \Delta_n(t) = 1. \tag{2.1}$$

Furthermore, for each $n \in \mathbb{Z}_+$,

$$\Delta_n(t) = \frac{1}{\lambda_n} \int_0^t H_n(u)du, \tag{2.2}$$

where $\lambda_n := 2^{-\frac{j}{2}-1}$ for $n = 2^j + k$ as above, and $\lambda_0 := 1$.

We work in $\Gamma(l^2(\mathbb{Z}_+))$. For each $n \in \mathbb{Z}_+$, let

$$e_n = (0, \ldots, 0, \overset{(n)}{1}, 0, \ldots,),$$

so $(e_n, n \in \mathbb{Z}_+)$ is an orthonormal basis for $l^2(\mathbb{Z}_+)$. Hence for each $g \in l^2(\mathbb{Z}_+)$,

$$g = \sum_{n=0}^{\infty} g_n e_n,$$

where $g_n := \langle g, e_n \rangle$.

We define $a_n := a(e_n)$ for each $n \in \mathbb{Z}_+$, then we have the canonical commutation relations:-

$$[a_m, a_n] = [a_n^\dagger, a_m^\dagger] = 0, [a_n, a_m^\dagger] = \delta_{mn},$$

for each $m, n \in \mathbb{Z}_+$. For all $f \in l^2(\mathbb{Z}_+), \psi(f)$ will denote the corresponding exponential vector. From now on we fix $H = \Gamma(l^2(\mathbb{Z}_+)), \mathcal{D}$ to be the linear span of the exponential vectors and ψ_0 to be the vacuum vector.

The following is well-known "Fock-law".

Proposition 2.1 *1.* $(\{a_n, n \in \mathbb{Z}_+\}, \psi_0)$ *is a Gaussian SHS. In particular all the finite dimensional distributions are i.i.d. Gaussian.*

2.

$$||a_n \psi(g)|| = |g_n| e^{\frac{||g||^2}{2}},$$

for all $g \in l^2(\mathbb{Z}_+), n \in \mathbb{Z}_+$.

3.

$$||a_n^\dagger \psi(g)|| = (1 + |g_n|^2)^{\frac{1}{2}} e^{\frac{||g||^2}{2}},$$

for all $g \in l^2(\mathbb{Z}_+), n \in \mathbb{Z}_+$.

Indeed (2) follows directly from the eigenrelation $a(f)\psi(g) = \langle f, g \rangle$, and (3) also follows from this via the commutation relations.

Theorem 2.1 *The series*

$$Y(t)\psi := \sum_{n=0}^{\infty} \lambda_n \Delta_n(t) a_n \psi \quad \textit{and} \quad Y(t)^{\dagger}\psi := \sum_{n=0}^{\infty} \lambda_n \Delta_n(t) a_n^{\dagger} \psi \tag{2.3}$$

converge uniformly for each $\psi \in \mathcal{D}$. The linear operators $Y(t)$ and $Y(t)^{\dagger}$ which are so defined are closable with each $Y(t)^{\dagger} \subseteq Y(t)^*$.

Furthermore the maps from $[0,1]$ to $\Gamma(l^2(\mathbb{Z}_+))$ given by $t \to Y(t)\psi$ and $t \to Y(t)^{\dagger}\psi$ are continuous.

Proof. It is sufficient to take $\psi = \psi(g)$ for some $g \in l^2(\mathbb{Z}_+)$. For any given $0 \leq t \leq 1$, we have $\Delta_n(t) = 0$ except for one n in each interval of the form $[2^j, 2^{j+1})$. We write each such n in the form $2^j + k_n$ where $0 \leq k_n < 2^j$. Using proposition 2.1 (1), (2.1) and the Cauchy-Schwarz inequality, we have for sufficiently large $M \geq 2^J$ (say)

$$\begin{aligned}
\left|\left| \sum_{n=M}^{\infty} \lambda_n \Delta_n(t) a_n \psi \right|\right| &\leq \sum_{n=M}^{\infty} \lambda_n \Delta_n(t) ||a_n \psi|| \\
&= \sum_{n=M}^{\infty} \lambda_n \Delta_n(t) |g_n| e^{\frac{1}{2}||g||^2} \\
&= \frac{1}{2} e^{\frac{1}{2}||g||^2} \sum_{j=J}^{\infty} \sum_{k=0}^{2^{j-1}} 2^{-\frac{j}{2}} \Delta_{2^j+k}(t) |g_{2^j+k}| \\
&= \frac{1}{2} e^{\frac{1}{2}||g||^2} \sum_{j=J}^{\infty} 2^{-\frac{j}{2}} |g_{2^j+k_n}| \\
&\leq \frac{1}{2} e^{\frac{1}{2}||g||^2} \left(\sum_{j=J}^{\infty} 2^{-j} \right)^{\frac{1}{2}} \left(\sum_{j=J}^{\infty} |g_{2^j+k_n}|^2 \right)^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \quad \frac{1}{2}||g||e^{\frac{1}{2}||g||^2}\left(\sum_{j=J}^{\infty}2^{-j}\right)^{\frac{1}{2}} \\
&\to 0 \quad \text{as } J \to \infty
\end{aligned}
$$

Using similar arguments and proposition 2.1 (2), we obtain

$$
\begin{aligned}
\left|\left|\sum_{n=M}^{\infty}\lambda_n\Delta_n(t)a_n^{\dagger}\psi\right|\right| &\leq \frac{1}{2}e^{\frac{1}{2}||g||^2}\sum_{j=J}^{\infty}2^{-\frac{j}{2}}(1+|g_{2^j+k_n}|^2)^{\frac{1}{2}} \\
&\leq \frac{1}{2}e^{\frac{1}{2}||g||^2}\sum_{j=J}^{\infty}2^{-\frac{j}{2}}(1+|g_{2^j+k_n}|) \\
&\leq \frac{1}{2}e^{\frac{1}{2}||g||^2}\left[\sum_{j=J}^{\infty}2^{-\frac{j}{2}}+||g||\left(\sum_{j=J}^{\infty}2^{-j}\right)^{\frac{1}{2}}\right] \\
&\to 0 \quad \text{as } J \to \infty
\end{aligned}
$$

The closure and mutual adjointness follow from the easily verified fact that for each $0 \leq t \leq 1, \phi_1, \phi_2 \in \mathcal{D}$,

$$
\langle Y(t)^{\dagger}\phi_1, \phi_2\rangle = \langle \phi_1, Y(t)\phi_2\rangle,
$$

and the continuity follows by the uniform convergence of each series on $[0,1]$. □

Theorem 2.2 *$(\{Y(t), t \geq 0\}, \psi_0)$ is a quantum Brownian motion.*

Proof. For each $0 \leq s, t \leq 1$ on the domain $\mathcal{D}$, using (2.2) and Parseval's identity, we have

$$\begin{aligned}
[Y(s), Y(t)^{\dagger}] &= \sum_{m,n=0}^{\infty} \lambda_m \lambda_n \Delta_m(s) \Delta_n(t) [a_m, a_n^{\dagger}] \\
&= \sum_{m,n=0}^{\infty} \lambda_m \lambda_n \Delta_m(s) \Delta_n(t) \delta_{mn} I \\
&= \sum_{n=0}^{\infty} \lambda_n \Delta_n(s) . \lambda_n \Delta_n(t) I \\
&= \sum_{n=0}^{\infty} \left(\int_0^s H_n(u) du \right) . \left(\int_0^t H_n(u) du \right) I \\
&= \sum_{n=0}^{\infty} \langle 1_{[0,s]}, H_n \rangle \langle 1_{[0,t]}, H_n \rangle I \\
&= \langle 1_{[0,s]}, 1_{[0,t]} \rangle I = (s \wedge t) I,
\end{aligned}$$

as required. The other two commutation relations are immediate.

To establish Gaussianity, let $\mathcal{P}$ be a partition of $[0, 1]$ containing $n+1$ points and denote the associated operators $b_{j,1,\mathcal{P}}$ simply as $b_j (1 \leq j \leq n)$. It follows from the commutation relations that

$$[b_j, b_k] = [b_j^{\dagger}, b_k^{\dagger}] = 0, [b_j, b_k^{\dagger}] = \delta_{jk},$$

for each $1 \leq j, k \leq n$. To see that these are independent Gaussians, for all $\alpha_j \in \mathbb{C} (1 \leq j \leq n)$

$$\left\langle \psi_0, \exp \left\{ \left(\sum_{j=1}^{n} (\alpha_j b_j^{\dagger} - \bar{\alpha_j} b_j) \right)^c \right\} \psi_0 \right\rangle$$

$$= \left\langle \psi_0, \exp\left\{-\frac{1}{2}\sum_{j=1}^n |\alpha_j|^2\right\} \exp\left\{\sum_{j=1}^n \alpha_j b_j^\dagger\right\} \exp\left\{\sum_{j=1}^n \bar{\alpha_j} b_j\right\} \psi_0 \right\rangle$$

$$= \exp\left\{-\frac{1}{2}\sum_{j=1}^n |\alpha_j|^2\right\}. \qquad \square$$

We have used wavelets to construct a quantum Brownian motion on $[0,1]$. To extend the index set to the whole of $\mathbb{R}^+$ we work in the *countable* (KRP) or *partial* (RFS) or *incomplete* (JvN) tensor product of an infinite number of copies of $\Gamma(l^2(\mathbb{N})$ with respect to the stabilising sequence comprising the vacuum vector in each space. We construct a countable number of i.i.d copies of quantum Brownian motion on $[0,1]$ via the prescription

$$Y^{(n)}(t) = I \otimes \cdots \otimes I \otimes \overset{(n)}{Y(t)} \otimes I \otimes \cdots$$

with domain the ampliation of $\mathcal{D}$. We then define quantum Brownian motion on $\mathbb{R}^+$ by as follows. We take ψ_0 to be the infinite tensor product of vacuum vectors, the domain to be the incomplete algebraic tensor product of an infinite number of copies of $\mathcal{D}$ and the required operators $(A(t), t \geq 0)$ are given by

$$A(t) = \sum_{k=1}^n Y^{(k)}(1) + Y^{(n+1)}(t-n),$$

whenever $t \in (n, n+1]$.

Note The construction extends to non-Fock, fermion, free, Bozejko-Speicher and monotone Brownian motions.

For the classical version, see

J.M.Steele, *Stochastic Calculus and Financial Applications*, Springer-Verlag New York Inc. (2001)

3 Quantum Brownian Bridges

We return to boson probability.

We define a *quantum Brownian bridge* to be a Gaussian SHS $(U(t), t \in [0,1]), \psi_0)$ for which

(i) $U(0) = U(1) = 0$.

(ii) For all $s, t \in [0,1]$, on $\mathcal{D}$,

$$[U(s), U(t)] = [U(s)^{\dagger}, U(t)^{\dagger}] = 0, [U(s), U(t)^{\dagger}] = (s \wedge t)[1-(s \vee t)]I.$$

To construct a quantum Brownian bridge, let $(X(t), t \in [0,1]), \psi_0)$ be a quantum Brownian motion. Define

$$U(t) := X(t) - tX(1),$$

for each $t \in [0,1]$. This is a quantum Brownian bridge. Indeed (i) and (ii) are both trivial and Gaussianity follows from writing each $U(t) = (1-t)X(t) - t(X(1) - X(t))$ and observing that $\{X(u), 0 \leq u \leq t), \psi_0\}$ and $\{(X(1) - X(u), t \leq u \leq 1), \psi_0\}$ are independent Gaussian SHSs. From the Cockroft-Hudson theorem 1.1, it follows that all quantum Brownian bridges are unitarily equivalent to that obtained from the Fock Brownian motion in this manner.

Using the Wiener-Segal duality transformation, we see that for each $\theta \in [0, 2\pi), e^{i\theta}U(t) + e^{-i\theta}U(t)^{\dagger}$ is unitarily equivalent to a classical Brownian bridge in Wiener space. The cases $\theta = 0$ and $\theta = \frac{\pi}{2}$ are naturally associated to canonical position and momentum field operators.

The following results are easily established analogues of well-known classical results.

1. If $(\{U(t), t \in [0,1]\}, \psi_0)$ is a quantum Brownian bridge then so is $(\{U(1-t), t \in [0,1]\}, \psi_0)$.

2. There is a one-to one correspondence between quantum Brownian motions $(\{A(t), t \in [0,1]\}, \psi_0)$ and quantum Brownian bridges $(\{U(t), t \in [0,1]\}, \psi_0)$ on the same space given by

$$A(t) \to (t+1)U\left(\frac{t}{t+1}\right), \quad \text{for all } t \in \mathbb{R}^+,$$

$$U(t) \to (1-t)A\left(\frac{t}{1-t}\right), \quad \text{for all } t \in [0,1).$$

Returning to the wavelet expansions of theorems 2.1 and 2.2, it follows just as in the classical case that $(\{V(t), t \in [0,1]\}, \psi_0)$ is a quantum Brownian bridge, where each

$$V(t) := \sum_{n=1}^{\infty} \lambda_n \Delta_n(t) a_n.$$

To see this, observe that since each $\Delta_n(1) = 0$ for each $n \in \mathbb{N}, Y^{\#}(1) = a_0^{\#}$, hence

$$V(t)^{\#} = Y(t)^{\#} - \Delta_0(t)a_0^{\#} = Y(t)^{\#} - tY(1)^{\#}.$$

The following is a quantum version of a well-known classical example of a Brownian bridge, however the method of the proof is completely different. We work in $\Gamma(L^2(\mathbb{R}^+))$.

Theorem 3.1 *$(\{U(t), t \in [0,1]\}, \psi_0)$ is a quantum Brownian bridge where for each $0 \le t < 1$*

$$U(t) := (1-t)\int_0^t \frac{dA(u)}{1-u}.$$

Proof. Gaussianity follows from the properties of quantum Wiener integrals and it is sufficient to verify the non-trivial commutation relation. For $0 \le s \le t < 1$, we use

$$U(t)^{\dagger} = (1-t)\int_0^s \frac{dA^{\dagger}(u)}{1-u} + (1-t)\int_s^t \frac{dA^{\dagger}(u)}{1-u}.$$

Then using the canonical commutation relations:

$$\begin{aligned} [U(s), U(t)^{\dagger}] &= (1-s)(1-t)\left[\int_0^s \frac{dA(u)}{1-u}, \int_0^s \frac{dA^{\dagger}(u)}{1-u}\right] \\ &= (1-s)(1-t)\int_0^s \frac{du}{(1-u)^2} I \\ &= s(1-t)I. \qquad \square \end{aligned}$$

A straightforward application of the quantum Itô formula applied to the result of theorem 3.1 yields the quantum stochastic differential equation representation of a quantum Brownian bridge:

$$dU(t) = dA(t) - \frac{1}{1-t}U(t)dt,$$

for all $0 \le t < 1$.

To finish, here's a beautiful application of *classical* Brownian bridges:

Let $(b(t), t \in [0,1])$ be a Brownian bridge and define

$$R := \sqrt{\frac{2}{\pi}} \left\{ \sup_{t\in[0,1]} b(t) - \inf_{t\in[0,1]} b(t) \right\},$$

then for all $z \in \mathbb{C}$ define

$$\xi(z) := \mathbb{E}(R^z).$$

ξ is an entire analytic function. It satisfies the functional equation

$$\xi(z) = \xi(1-z).$$

More spectacularly,

$$\xi(z) = \frac{1}{2} z(1-z)\pi^{-\frac{z}{2}} \Gamma\left(\frac{1}{2}z\right) \zeta(z),$$

where ζ is the *Riemann zeta function.*

Some references for this:

D.Williams, Brownian motion and the Riemann zeta-function, *Disorder in Physical Systems* ed. G.R.Grimmett and D.J.A. Welsh, pp.361-72, Clarendon Press, Oxford (1990)

P.Biane, J.Pitman, M.Yor, Probability laws related to the Jacobi theta and Riemann zeta functions, and Brownian excursions, *Bull. Amer. Math. Soc.*, **38**, 435-67 (2001)

Challenge: Find a link between *quantum Brownian bridges* and the *Riemann zeta function.*

Also - classical Brownian bridges can be used to prove the Gauss-Bonnet-Chern theorem. Diffusion bridges are applied to prove the Atiyah-Singer Index theorem. What can you do with quantum versions ?