# Lévy-type Stochastic Integrals with Heavy Tails

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# 1 Regular Variation

Let $F$ be the distribution function of a non-negative random variable $F(x) = P(X \leqslant x)$. Tail $\overline{F}(x) = 1 - F(x) = P(X > x)$.

$X$ (or $F$) has *regular variation at infinity with index* $\alpha \geqslant 0$ - $F \in \mathcal{R}_{-\alpha}$ if

$$\lim_{x \to \infty} \frac{\overline{F}(cx)}{\overline{F}(x)} = c^{-\alpha}, \quad \text{for all } c > 0,$$

$$\text{i.e. } \overline{F}(x) = x^{-\alpha} L(x), L \in \mathcal{R}_0.$$

Regular variation is important in probability theory as:-

- Subexponential, $\lim_{x \to \infty} \dfrac{\overline{F * F}(x)}{\overline{F}(x)} = 2$

  $$\Rightarrow P(X_1 + X_2 > x) \sim P(\max\{X_1, X_2\} > x),$$

  where $X_1, X_2$ are independent copies of $X$.

- Domains of attraction

  - in CLT for stable laws (DNA)

  - in extreme value theory for Fréchet distribution

- Applications to e.g. insurance risk, finance, communication systems:-

  Large Deviations for Heavy tails: - rare events are caused by the smallest possible number of individual factors.

P.Embrechts, C.Klüppelberg, T.Mikosch, *Modelling Extremal Events for Insurance and Finance*, Springer-Verlag, Berlin, Heidelberg (1997)

Left tails: Let $g : (-\infty, 0) \to \mathbb{R}^+$ and define $\tilde{g} : (0, \infty) \to \mathbb{R}^+$ by $\tilde{g}(x) = g(-x)$, for all $x > 0$. $g \in \mathcal{L}_\alpha$ if and only if $\tilde{g} \in \mathcal{R}_\alpha$.

## 2 Lévy Processes

$(X(t), t \geqslant 0)$ has stationary and independent increments, $X(0) = 0$ (a.s.), stochastically continuous and càdlàg paths.

Write $F_t = F_{X(t)}$

**The Lévy-Itô decomposition**

$$\begin{aligned} X(t) &= bt + \sigma B(t) + \int_{|x|<1} x\tilde{N}(t, dx) + \int_{|x|\geqslant 1} xN(t, dx) \\ &= \text{drift} + \text{diffusion} + \text{small jumps} + \text{large jumps} \end{aligned}$$

where

- $b \in \mathbb{R}, \sigma \geqslant 0$.
- $B$ is a standard Brownian motion.
- $N$ is a Poisson random measure on $\mathbb{R}^+ \times (\mathbb{R} - \{0\})$ with intensity measure Leb $\otimes \nu$

$$N(t, A) = \#\{0 \leqslant s \leqslant t; \Delta X(s) \in A\}.$$

- $\tilde{N}$ is the *compensator* - $\tilde{N}(dt, dx) = N(dt, dx) - \nu(dx)dt$.
- $\nu$ is a *Lévy measure* $\int_{\mathbb{R}-\{0\}}(|x|^2 \wedge 1)\nu(dx) < \infty$.

**Tail Equivalence**

$$\overline{F_t} \in \mathcal{R}_{-\alpha} \Leftrightarrow \overline{\nu} \in \mathcal{R}_{-\alpha} \Rightarrow \lim_{x\to\infty} \frac{\overline{F_t}(x)}{t\overline{\nu}(x)} = 1$$

- Feller (1971), Embrechts and Goldie (1981),

(multivariate case - Hult and Lindskog (2002)).

## 3 Stock Price Models

Stock price evolution $(S(t), t \geqslant 0)$.

Black-Scholes model $S(t) = S_0 \exp\left\{\sigma B(t) + \left(\mu - \frac{1}{2}\sigma^2 t\right)\right\}$.

Defects of Black-Scholes model:-

Empirical evidence suggests

- Non-Gaussian log returns (kurtosis $> 3$).
- Non-constant volatility.

Lévy models $S(t) = S_0 \exp X(t)$

e.g. $X$ hyperbolic, generalised inverse Gaussian, Meixner process. Geman, Madan, Yor - small jumps "infinite activity", large jumps "finite activity".

see W.Schoutens, *Lévy Processes in Finance: Pricing Financial Derivatives*, Wiley (2003)

Some empirical evidence for heavy tails with $2 \leqslant \alpha \leqslant 4$. Perhaps we should take $X$ to be a more complicated semimartingale with jumps and heavy tails ?

## 4 Lévy-Type Stochastic Integrals

$(\Omega, \mathcal{F}, P)$. Filtration $(\mathcal{F}_t, t \geqslant 0)$. $\mathcal{P}$ predictable $\sigma$-algebra.

$$E := \{x \in \mathbb{R}; 0 < |x| < 1\} \quad , \quad E^c = \{x \in \mathbb{R}; |x| > 1\}.$$

$(F, G, H, K)$ a quadruple

- $F = (F(t), t \geqslant 0), G = (G(t), t \geqslant 0)$ are predictable
- $H = (H(t,x), t \geqslant 0, x \in E)$ is $\mathcal{P} \otimes \mathcal{B}(E)$ measurable.
- $K = (K(t,x), t \geqslant 0, x \in E^c)$ is $\mathcal{P} \otimes \mathcal{B}(E^c)$ measurable.

Assume $\displaystyle\int_0^t \left( |F(s)| + |G(s)|^2 + \int_E |H(s,x)|^2 \nu(dx) \right) ds < \infty$ a.s..

*Lévy-type stochastic integral* $M = (M(t), t \geqslant 0)$, for each $t \geqslant 0$,

$$\begin{aligned} M(t) &:= \int_0^t F(s)ds + \int_0^t G(s)dB(s) + \int_0^t \int_E H(s,x)\tilde{N}(ds,dx) \\ &+ \int_0^t \int_{E^c} K(s,x)N(ds,dx) \\ &:= I_1^F(t) + I_2^G(t) + I_3^H(t) + I_4^K(t) \qquad (4.1) \\ &= \text{drift} + \text{diffusion} + \text{small jumps} + \text{large jumps} \qquad (4.2) \end{aligned}$$

$M$ is a semimartingale.

- $I_2^G$ and $I_3^H$ are local martingales.
- $I_1^F$ and $I_4^K$ are processes of finite variation.
- $I_4^K(t)$ is a (random) finite sum - $I_4^K(t) = \sum_{0 \leqslant s \leqslant t} K(s, \Delta Y(s))$, where $Y(s) = \int_{|x| \geqslant 1} x N(s, dx)$.

Reference - D.Applebaum *Lévy Processes and Stochastic Calculus*, Cambridge University Press (2004)

## 5 Moment Estimates

Aim: To find sufficient conditions for $M$ to have regular variation.

Strategy: By imposing suitable moment conditions on $F, G$ and $H$, show that the right tails of $I_1^F, I_2^G$ and $I_3^H$ decay faster than any negative power of $x$.

Fix $T \geqslant 0, 0 < t \leqslant T$

Test case: $I_2^G(t) = \int_0^t G(s)dB(s)$.

If $0 \leqslant \alpha < 2$, we assume that $\int_0^T \mathbb{E}(|G(s)|^2)ds < \infty$.

Itô's isometry yields

$$\mathbb{E}(|I_2^G(t)|^2) = \int_0^t \mathbb{E}(|G(s)|^2)ds.$$

Markov's inequality yields

$$P(|I_2^G(t)| \geqslant \lambda) \leqslant \frac{\int_0^t \mathbb{E}(|G(s)|^2)ds}{\lambda^2},$$

hence for any $L \in \mathcal{R}_0$

$$\limsup_{\lambda \to \infty} \frac{P(|I_2^G(t)| \geqslant \lambda)}{\lambda^{-\alpha} L(\lambda)} = 0.$$

If $\alpha \geqslant 2$, assume that $\int_0^t \mathbb{E}(|G(s)|^{\alpha+\epsilon})ds < \infty$, for some $\epsilon > 0$.
As above, we have

$$P(|I_2^G(t)| \geqslant \lambda) \leqslant \frac{\mathbb{E}(|I_2^G(t)|^{\alpha+\epsilon})}{\lambda^{\alpha+\epsilon}}.$$

Using Burkholder and Hölder's inequalities, we obtain

$$\begin{aligned}
\mathbb{E}(|I_2^G(t)|^{\alpha+\epsilon}) &\leqslant C_1(\alpha,\epsilon)\mathbb{E}([I_2^G, I_2^G](t)^{\frac{\alpha+\epsilon}{2}}) \\
&= C_1(\alpha,\epsilon)\mathbb{E}\left[\left(\int_0^t |G(s)|^2 ds\right)^{\frac{\alpha+\epsilon}{2}}\right] \\
&\leqslant C_1(\alpha,\epsilon) t^{\frac{\alpha+\epsilon-2}{2}} \int_0^t \mathbb{E}(|G(s)|^{\alpha+\epsilon})ds,
\end{aligned}$$

and we are finished.
Similar arguments are used to deal with $I_1^F$ and $I_3^H$.
For $I_3^H$ use recent estimate due to H.Kunita: for each $p \geqslant 2$,

$$\begin{aligned}
\mathbb{E}(|I_3^H(t)|^p) &\leqslant C_2(p)\left\{\int_0^t \int_E \mathbb{E}(|H(s,x)|^p)\nu(dx)ds\right. \\
&+ \left.\mathbb{E}\left[\left(\int_0^t \int_E |H(s,x)|^2 \nu(dx)ds\right)^{\frac{p}{2}}\right]\right\},
\end{aligned}$$

for all $0 \leqslant t \leqslant T$, where $C_2(p) > 0$.

## 6 Assumptions

1. For all $x \in E^c, K(t,x) = K(t)f(x)$,

   where $\inf_{0 \leqslant s \leqslant t} K(s) > 0$ for all $t > 0, f(x) \geqslant 0$.

2. $f_+ := f1_{\{x \geqslant 1\}} \in \mathcal{R}_\beta$ for some $\beta > 0$ and is non-decreasing with $\lim_{x\to\infty} f_+(x) = \infty$.

   $f_- := f1_{\{x \leqslant -1\}} \in \mathcal{L}_\delta$ for some $\delta > 0$ and is non-increasing with $\lim_{x\to-\infty} f_-(x) = \infty$.

3. $K$ is cáglád and independent of $N$. For each $0 \leqslant t \leqslant T$, there exists $\epsilon(t) > 0$ such that $\mathbb{E}(\overline{K}(t)^{\rho+\epsilon(t)}) < \infty$, for some fixed $\rho > 0$.

   $$[\overline{K}(t) = \sup_{0 \leqslant s \leqslant t} K(s), \quad \underline{K}(t) = \inf_{0 \leqslant s \leqslant t} K(s) \quad \text{for each } t \geqslant 0].$$

4. $\nu((-\infty, \lambda)) \in \mathcal{L}_{-\gamma}$ and $\nu((\lambda, \infty) \in \mathcal{R}_{-\alpha}$, where $\alpha, \gamma \geqslant 0$.

5. *Asymptotic independence*

   For all $a \in \mathbb{R}$,

   $P(M(t) - I_4^K(t) > a | I_4^K(t) > b) \sim P(M(t) - I_4^K(t) > a)$ as $b \to \infty$.

## 7 The Associated Compound Poisson Process

Define $Z_f(t) = \int_{|x|\geqslant 1} f(x)N(t,dx)$

**Proposition 7.1** *$\overline{F_{Z_f(t)}} \in \mathcal{R}_{-\rho}$, where $\rho = \min\left\{\frac{\alpha}{\beta}, \frac{\gamma}{\delta}\right\}$.*

*Proof*

$$\begin{aligned} Z_f(t) &= \int_{x\leqslant -1} f(x)N(t,dx) + \int_{x\geqslant 1} f(x)N(t,dx) \\ &:= Z_f^+(t) + Z_f^-(t). \end{aligned}$$

$Z_f^+$ and $Z_f^-$ are independent compound Poisson processes with Lévy measures $\nu \circ f_+^{-1}$ and $\nu \circ f_-^{-1}$, respectively.

It follows that $\overline{\nu_{f_+}} \in \mathcal{R}_{-\frac{\alpha}{\beta}}$ and $\overline{\nu_{f_-}} \in \mathcal{R}_{-\frac{\gamma}{\delta}}$,

[see Proposition 0.8 in S.Resnick, *Extreme Values, Regular Variation and Point Processes*, Springer-Verlag, New York (1987)]

By tail equivalence, $\overline{F_{Z_f^+(t)}} \in \mathcal{R}_{-\frac{\alpha}{\beta}}$ and $\overline{F_{Z_f^-(t)}} \in \mathcal{R}_{-\frac{\gamma}{\delta}}$.

Result follows by fact that:

If $X$ and $Y$ are independent random variables, with $\overline{F_X} \in \mathcal{R}_{-a}$ and $\overline{F_Y} \in \mathcal{R}_{-b}$, then $\overline{F_{X+Y}} \in \mathcal{R}_{-\min\{a,b\}}$.

# 8 Regular Variation of the Process $I_4^K(t) = \int_0^t \int_{|x| \geqslant 1} K(s,x)N(ds,dx)$

Fact: (Breiman) If $\overline{F_X} \in \mathcal{R}_{-\alpha}$ and $Y > 0$ is independent of $X$ with $\mathbb{E}(Y^{\alpha+\epsilon}) < \infty$, for some $\epsilon > 0$ then $\overline{F_{XY}} \in \mathcal{R}_{-\alpha}$.

[see section 4.2 in S.Resnick, Point processes, regular variation and weak convergence, *Adv. Appl. Prob.* **18**, 66-138 (1986)]

**Theorem 8.1** $\overline{F_{I_4^K(t)}} \in \mathcal{R}_{-\rho}$ *for each* $0 < t \leqslant T$.

*Proof.* Using assumption 3, we obtain

$$\begin{aligned}
1 &= \lim_{\lambda\to\infty} \frac{P(\underline{K}(t)Z_f(t) > \lambda)}{\lambda^{-\rho}L(\rho)} \\
&\leqslant \liminf_{\lambda\to\infty} \frac{P(I_4^K(t) > \lambda)}{\lambda^{-\rho}L(\rho)} \leqslant \limsup_{\lambda\to\infty} \frac{P(I_4^K(t) > \lambda)}{\lambda^{-\rho}L(\rho)} \\
&\leqslant \lim_{\lambda\to\infty} \frac{P(\overline{K}(t)Z_f(t) > \lambda)}{\lambda^{-\rho}L(\rho)} = 1,
\end{aligned}$$

and the required result follows. □

e.g. Take each $K(t) = g(B(t))$, where $B = (B(t), t \geqslant 0)$ is a standard Brownian motion and $g : \mathbb{R} \to (0,\infty)$ is continuous, convex and polynomially bounded.

## 9 The Main Theorem

**Theorem 9.1** *Let $M = (M(t), 0 \leqslant t \leqslant T)$ be a Lévy-type stochastic integral of the form satisfying the conditions (1) to (5). Further assume the following:-*

- *If $0 \leqslant \rho \leqslant 1$,*

$$\int_0^T \left[\mathbb{E}\left(|F(s)| + |G(s)|^2 + \int_E |H(s,x)|^2 \nu(dx)\right)\right] ds < \infty.$$

- *If $1 \leqslant \rho < 2$, for some $\epsilon > 0$,*

$$\int_0^T \left[\mathbb{E}\left(|F(s)|^{\rho+\epsilon} + |G(s)|^2 + \int_E |H(s,x)|^2 \nu(dx)\right)\right] ds < \infty.$$

- *If $\rho \geqslant 2$, for some $\delta_1, \delta_2, \delta_3 > 0$,*

$$\int_0^T \left[\mathbb{E}\left(|F(s)|^{\rho+\delta_1} + |G(s)|^{\rho+\delta_2} + \int_E |H(s,x)|^{\rho+\delta_3} \nu(dx)\right)\right] ds < \infty,$$

*if $\nu(E) < \infty$ or,*

$$\int_0^T \left[\mathbb{E}\left(|F(s)|^{\rho+\delta_1} + |G(s)|^{\rho+\delta_2} + \int_E |H(s,x)|^{\rho+\delta_3} \nu(dx)\right)\right] ds + \mathbb{E}\left[\left(\int_0^T \int_E |H(s,x)|^2 \nu(dx) ds\right)^{\frac{\rho+\delta_3}{2}}\right] < \infty,$$

*if $\nu(E) = \infty$.*

*Then $\overline{F_{M(t)}} \in \mathcal{R}_{-\rho}$ for each $0 < t \leqslant T$.*

*Proof of the theorem*

Let $N(t) = M(t) - I_4^K(t)$.

For each $0 < \eta < 1$,

$$\begin{aligned} P(M(t) > \lambda) &\leqslant P(I_4^K(t) > (1-\eta)\lambda) + P(N(t) > (1-\eta)\lambda) \\ &+ P(I_4^K(t) \geqslant \eta\lambda, N(t) \geqslant \eta\lambda) \\ &\leqslant P(I_4^K(t) > (1-\eta)\lambda) + P(|N(t)| \geqslant (1-\eta)\lambda) \\ &+ P(|N(t)| \geqslant \eta\lambda). \end{aligned}$$

Now for any $\kappa > 0$,

$$P(|N(t)| \geqslant \kappa) \leqslant P\left(|I_1^F(t)| \geqslant \frac{\kappa}{3}\right) + P\left(|I_2^G(t)| \geqslant \frac{\kappa}{3}\right) + P\left(|I_3^H(t)| \geqslant \frac{\kappa}{3}\right).$$

Moment estimates ensure that

$$\lim_{\lambda\to\infty} \frac{P(|N(t)| \geqslant (1-\eta)\lambda) + P(|N(t)| \geqslant \eta\lambda)}{\lambda^{-\rho} L(\lambda)} = 0,$$

for any $L \in \mathcal{R}_0$.

Hence $\limsup_{\lambda\to\infty} \frac{P(M(t) > \lambda)}{\lambda^{-\rho} L(\lambda)} \leqslant \lim_{\lambda\to\infty} \frac{P(I_4^K(t) > (1-\eta)\lambda)}{\lambda^{-\rho} L(\lambda)} = (1-\eta)^{-\rho}$.

Now take limits as $\eta \downarrow 0$, to obtain

$$\limsup_{\lambda\to\infty} \frac{P(M(t) > \lambda)}{\lambda^{-\rho} L(\lambda)} \leqslant 1.$$

For the reverse inequality, fix $C > 0$, then

$$\begin{aligned} P(M(t) > \lambda) &\geqslant P(N(t) > -C, I_4^K(t) > \lambda + C) \\ &= P(N(t) > -C | I_4^K(t) > \lambda + C) P(I_4^K(t) > \lambda + C). \end{aligned}$$

By the asymptotic independence assumption

$$P(N(t) > -C | I_4^K(t) > \lambda + C) \sim P(N(t) > -C), \text{ as } \lambda \to \infty.$$

By the representation theorem for slowly varying functions,

$$P(I_4^K(t) > \lambda + C) \sim P(I_4^K(t) > \lambda), \text{ as } \lambda \to \infty.$$

Hence deduce that

$$\liminf_{\lambda \to \infty} \frac{P(M(t) > \lambda)}{\lambda^{-\rho} L(\lambda)} \geqslant P(N(t) > -C).$$

Now take limits as $C \to \infty$, to obtain

$$\liminf_{\lambda \to \infty} \frac{P(M(t) > \lambda)}{\lambda^{-\rho} L(\lambda)} \geqslant 1,$$

and the result follows. □

## 10 Extensions

- Similar results hold in *multivariate* case - use recent ideas of

  F.Lindskog, *Multivariate Extremes and Regular Variation for Stochastic Processes*, Diss. ETH No.15319 (2004)

- Extensions to more complicated classes of semimartingales ?
- What about subexponentiality ?
- Applications to finance ?