Lectures on Lévy Processes and Stochastic Calculus, Braunschweig, Lecture 3: The Lévy-Itô Decomposition

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July 22nd - 24th 2010

Filtrations, Markov Processes and Martingales

We recall the probability space $(\Omega, \mathcal{F}, P)$ which underlies our investigations. $\mathcal{F}$ contains all possible events in Ω.
When we introduce the arrow of time, its convenient to be able to consider only those events which can occur up to and including time t. We denote by $\mathcal{F}_t$ this sub-σ-algebra of $\mathcal{F}$. To be able to consider all time instants on an equal footing, we define a *filtration* to be an increasing family $(\mathcal{F}_t, t \geq 0)$ of sub-σ-algebras of $\mathcal{F}$, , i.e.

$$0 \leq s \leq t < \infty \Rightarrow \mathcal{F}_s \subseteq \mathcal{F}_t.$$

A stochastic process $X = (X(t), t \geq 0)$ is *adapted* to the given filtration if each $X(t)$ is $\mathcal{F}_t$-measurable.

e.g. any process is adapted to its *natural filtration*, $\mathcal{F}_t^X = \sigma\{X(s); 0 \leq s \leq t\}$.

An adapted process $X = (X(t), t \geq 0)$ is a *Markov process* if for all $f \in B_b(\mathbb{R}^d), 0 \leq s \leq t < \infty$,

$$\mathbb{E}(f(X(t))|\mathcal{F}_s) = \mathbb{E}(f(X(t))|X(s)) \quad (a.s.). \tag{0.1}$$

(i.e. "past" and "future" are independent, given the present).

The *transition probabilities* of a Markov process are $p_{s,t}(x, A) = P(X(t) \in A|X(s) = x)$,
i.e. the probability that the process is in the Borel set A at time t given that it is at the point x at the earlier time s.

Theorem

If X is a Lévy process (adapted to its own natural filtration) wherein each $X(t)$ has law q_t, then it is a Markov process with transition probabilities $p_{s,t}(x, A) = q_{t-s}(A - x)$.

Proof. This essentially follows from

$$\begin{aligned}\mathbb{E}(f(X(t))|\mathcal{F}_s) &= \mathbb{E}(f(X(s) + X(t) - X(s))|\mathcal{F}_s) \\ &= \int_{\mathbb{R}^d} f(X(s) + y) q_{t-s}(dy).\end{aligned}$$

□

Now let X be an adapted process defined on a filtered probability space which also satisfies the integrability requirement $\mathbb{E}(|X(t)|) < \infty$ for all $t \geq 0$.
We say that it is a *martingale* if for all $0 \leq s < t < \infty$,

$$\mathbb{E}(X(t)|\mathcal{F}_s) = X(s) \quad \text{a.s.}$$

Note that if X is a martingale, then the map $t \to \mathbb{E}(X(t))$ is constant.

An adapted Lévy process with zero mean is a martingale (with respect to its natural filtration)
since in this case, for $0 \leq s \leq t < \infty$ and using the convenient notation $\mathbb{E}_s(\cdot) := \mathbb{E}(\cdot|\mathcal{F}_s)$:

$$\begin{aligned} \mathbb{E}_s(X(t)) &= \mathbb{E}_s(X(s) + X(t) - X(s)) \\ &= X(s) + \mathbb{E}(X(t) - X(s)) = X(s) \end{aligned}$$

Although there is no good reason why a generic Lévy process should be a martingale (or even have finite mean), there are some important examples:

e.g. the processes whose values at time t are

- $\sigma B(t)$ where $B(t)$ is a standard Brownian motion, and σ is an $r \times d$ matrix.
- $\tilde{N}(t)$ where $\tilde{N}$ is a compensated Poisson process with intensity λ.

Some important martingales associated to Lévy processes include:

- $\exp\{i(u, X(t)) - t\eta(u)\}$, where $u \in \mathbb{R}^d$ is fixed.
- $|\sigma B(t)|^2 - \text{tr}(A)t$ where $A = \sigma^T\sigma$.
- $\tilde{N}(t)^2 - \lambda t$.

Càdlàg Paths

A function $f : \mathbb{R}^+ \to \mathbb{R}^d$ is *càdlàg* if it is *continue à droite et limité à gauche*, i.e. right continuous with left limits. Such a function has only jump discontinuities.
Define $f(t-) = \lim_{s \uparrow t} f(s)$ and $\Delta f(t) = f(t) - f(t-)$. If f is càdlàg, $\{0 \leq t \leq T, \Delta f(t) \neq 0\}$ is at most countable.

If the filtration satisfies the "usual hypotheses" of right continuity and completion, then every Lévy process has a càdlàg modification which is itself a Lévy process.

From now on, we will always make the following assumptions:-

- $(\Omega, \mathcal{F}, P)$ will be a fixed probability space equipped with a filtration $(\mathcal{F}_t, t \geq 0)$ which satisfies the "usual hypotheses".
- Every Lévy process $X = (X(t), t \geq 0)$ will be assumed to be $\mathcal{F}_t$-adapted and have càdlàg sample paths.
- $X(t) - X(s)$ is independent of $\mathcal{F}_s$ for all $0 \leq s < t < \infty$.

The Jumps of A Lévy Process - Poisson Random Measures

The *jump process* $\Delta X = (\Delta X(t), t \geq 0)$ associated to a Lévy process is defined by

$$\Delta X(t) = X(t) - X(t-),$$

for each $t \geq 0$.

Theorem

If N is a Lévy process which is increasing (a.s.) and is such that $(\Delta N(t), t \geq 0)$ takes values in $\{0, 1\}$, then N is a Poisson process.

Proof. Define a sequence of stopping times recursively by $T_0 = 0$ and $T_n = \inf\{t > T_{n-1}; N(t + T_{n-1}) - N(T_{n-1})) \neq 0\}$ for each $n \in \mathbb{N}$. It follows from (L2) that the sequence $(T_1, T_2 - T_1, \ldots, T_n - T_{n-1}, \ldots)$ is i.i.d.

By (L2) again, we have for each $s, t \geq 0$,

$$\begin{aligned} P(T_1 > s + t) &= P(N(s) = 0, N(t + s) - N(s) = 0) \\ &= P(T_1 > s)P(T_1 > t) \end{aligned}$$

From the fact that N is increasing (a.s.), it follows easily that the map $t \to P(T_1 > t)$ is decreasing and by a straightforward application of stochastic continuity (L3) we find that the map $t \to P(T_1 > t)$ is continuous at $t = 0$. Hence there exists $\lambda > 0$ such that $P(T_1 > t) = e^{-\lambda t}$ for each $t \geq 0$.

So T_1 has an exponential distribution with parameter λ and

$$P(N(t) = 0) = P(T_1 > t) = e^{-\lambda t},$$

for each $t \geq 0$.
Now assume as an inductive hypothesis that $P(N(t) = n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}$, then

$$P(N(t) = n + 1) = P(T_{n+2} > t, T_{n+1} \leq t) = P(T_{n+2} > t) - P(T_{n+1} > t).$$

$$\text{But } T_{n+1} = T_1 + (T_2 - T_1) + \cdots + (T_{n+1} - T_n)$$

is the sum of $(n+1)$ i.i.d. exponential random variables, and so has a gamma distribution with density $f_{T_{n+1}}(s) = e^{-\lambda s}\dfrac{\lambda^{n+1}s^n}{n!}$ for $s > 0$.
The required result follows on integration. $\square$

The following result shows that ΔX is not a straightforward process to analyse.

Lemma

If X is a Lévy process, then for fixed $t > 0, \Delta X(t) = 0$ (a.s.).

Proof. Let $(t(n), n \in \mathbb{N})$ be a sequence in $\mathbb{R}^+$ with $t(n) \uparrow t$ as $n \to \infty$, then since X has càdlàg paths, $\lim_{n\to\infty} X(t(n)) = X(t-)$.However, by (L3) the sequence $(X(t(n)), n \in \mathbb{N})$ converges in probability to $X(t)$, and so has a subsequence which converges almost surely to $X(t)$. The result follows by uniqueness of limits. □

Much of the analytic difficulty in manipulating Lévy processes arises from the fact that it is possible for them to have

$$\sum_{0 \leq s \leq t} |\Delta X(s)| = \infty \quad \text{a.s.}$$

and the way in which these difficulties is overcome exploits the fact that we always have

$$\sum_{0 \leq s \leq t} |\Delta X(s)|^2 < \infty \quad \text{a.s.}$$

We will gain more insight into these ideas as the discussion progresses.

Rather than exploring ΔX itself further, we will find it more profitable to count jumps of specified size. More precisely, let $0 \leq t < \infty$ and $A \in \mathcal{B}(\mathbb{R}^d - \{0\})$. Define

$$\begin{aligned} N(t, A) &= \#\{0 \leq s \leq t; \Delta X(s) \in A\} \\ &= \sum_{0 \leq s \leq t} \mathbf{1}_A(\Delta X(s)). \end{aligned}$$

Note that for each $\omega \in \Omega, t \geq 0$, the set function $A \to N(t, A)(\omega)$ is a counting measure on $\mathcal{B}(\mathbb{R}^d - \{0\})$ and hence

$$\mathbb{E}(N(t, A)) = \int N(t, A)(\omega) dP(\omega)$$

is a Borel measure on $\mathcal{B}(\mathbb{R}^d - \{0\})$. We write $\mu(\cdot) = \mathbb{E}(N(1, \cdot))$.

We say that $A \in \mathcal{B}(\mathbb{R}^d - \{0\})$ is *bounded below* if $0 \notin \bar{A}$.

Lemma

If A is bounded below, then $N(t, A) < \infty$ (a.s.) for all $t \geq 0$.

Proof. Define a sequence of stopping times $(T_n^A, n \in \mathbb{N})$ by $T_1^A = \inf\{t > 0; \Delta X(t) \in A\}$, and for $n > 1, T_n^A = \inf\{t > T_{n-1}^A; \Delta X(t) \in A\}$.
Since X has càdlàg paths, we have $T_1^A > 0$ (a.s.) and $\lim_{n\to\infty} T_n^A = \infty$ (a.s.).
Indeed suppose that $T_1^A = 0$ with non-zero probability and let $\mathcal{N} = \{\omega \in \Omega : T_1^A \neq 0\}$. Assume that $\omega \in \Omega - \mathcal{N}$. Then given any $u > 0$, we can find $0 < \delta, \delta' < u$ and $\epsilon > 0$ such that $|X(\delta)(\omega) - X(\delta')(\omega)| > \epsilon$ and this contradicts the (almost sure) right continuity of $X(\cdot)(\omega)$ at the origin.

Similarly, we assume that $\lim_{n\to\infty} T_n^A = T^A < \infty$ with non-zero probability and define $\mathcal{M} = \{\omega \in \Omega : \lim_{n\to\infty} T_n^A = \infty\}$. If $\omega \in \Omega - \mathcal{M}$ then we obtain a contradiction with the fact that X has a left limit (almost surely) at $T^A(\omega)$.
Hence, for each $t \geq 0$,

$$N(t, A) = \sum_{n \in \mathbb{N}} \mathbf{1}_{\{T_n^A \leq t\}} < \infty \text{ a.s.} \qquad \square$$

Be aware that if A fails to be bounded below, then this lemma may no longer hold, because of the accumulation of large numbers of small jumps.
The following result should at least be plausible, given Theorem 2 and Lemma 4.

Theorem

1. *If A is bounded below, then $(N(t, A), t \geq 0)$ is a Poisson process with intensity $\mu(A)$.*
2. *If $A_1, \ldots, A_m \in \mathcal{B}(\mathbb{R}^d - \{0\})$ are disjoint, then the random variables $N(t, A_1), \ldots, N(t, A_m)$ are independent.*

It follows immediately that $\mu(A) < \infty$ whenever A is bounded below, hence the measure μ is σ-finite.

The main properties of N, which we will use extensively in the sequel, are summarised below:-.

1. For each $t > 0, \omega \in \Omega, N(t, \cdot)(\omega)$ is a counting measure on $\mathcal{B}(\mathbb{R}^d - \{0\})$.
2. For each A bounded below, $(N(t, A), t \geq 0)$ is a Poisson process with intensity $\mu(A) = \mathbb{E}(N(1, A))$.
3. The *compensator* $(\tilde{N}(t, A), t \geq 0)$ is a martingale-valued measure where $\tilde{N}(t, A) = N(t, A) - t\mu(A)$, for A bounded below, i.e. For fixed A bounded below, $(\tilde{N}(t, A), t \geq 0)$ is a martingale.

Poisson Integration

Let f be a Borel measurable function from $\mathbb{R}^d$ to $\mathbb{R}^d$ and let A be bounded below, then for each $t > 0, \omega \in \Omega$, we may define the *Poisson integral* of f as a random finite sum by

$$\int_A f(x) N(t, dx)(\omega) := \sum_{x \in A} f(x) N(t, \{x\})(\omega).$$

Note that each $\int_A f(x) N(t, dx)$ is an $\mathbb{R}^d$-valued random variable and gives rise to a càdlàg stochastic process, as we vary t.
Now since $N(t, \{x\}) \neq 0 \Leftrightarrow \Delta X(u) = x$ for at least one $0 \leq u \leq t$, we have

$$\int_A f(x) N(t, dx) = \sum_{0 \leq u \leq t} f(\Delta X(u)) \mathbf{1}_A(\Delta X(u)). \qquad (0.2)$$

In the sequel, we will sometimes use μ_A to denote the restriction to A of the measure μ. in the following theorem, *Var* stands for variance.

Theorem

Let A be bounded below, then

1. $(\int_A f(x)N(t,dx), t \geq 0)$ *is a compound Poisson process, with characteristic function*

$$\mathbb{E}\left(\exp\left\{i\left(u, \int_A f(x)N(t,dx)\right)\right\}\right) = \exp\left[t\int_{\mathbb{R}^d}(e^{i(u,x)} - 1)\mu_{f,A}(dx)\right]$$

for each $u \in \mathbb{R}^d$*, where* $\mu_{f,A}(B) := \mu(A \cap f^{-1}(B))$*, for each* $B \in \mathcal{B}(\mathbb{R}^d)$.

2. *If* $f \in L^1(A, \mu_A)$*, then*

$$\mathbb{E}\left(\int_A f(x)N(t,dx)\right) = t\int_A f(x)\mu(dx).$$

Theorem

3 *If* $f \in L^2(A, \mu_A)$*, then*

$$Var\left(\left|\int_A f(x)N(t,dx)\right|\right) = t\int_A |f(x)|^2\mu(dx).$$

Proof. - part of it!
1) For simplicity, we will prove this result in the case where $f \in L^1(A, \mu_A)$. First let f be a simple function and write $f = \sum_{j=1}^n c_j \mathbf{1}_{A_j}$ where each $c_j \in \mathbb{R}^d$. We can assume, without loss of generality, that the A_j's are disjoint Borel subsets of A.

By Theorem 5, we find that

$$\begin{aligned}
\mathbb{E}\left(\exp\left\{i\left(u, \int_A f(x)N(t,dx)\right)\right\}\right) &= \mathbb{E}\left(\exp\left\{i\left(u, \sum_{j=1}^n c_j N(t,A_j)\right)\right\}\right) \\
&= \prod_{j=1}^n \mathbb{E}\left(\exp\left\{i\left(u, c_j N(t,A_j)\right)\right\}\right) \\
&= \prod_{j=1}^n \exp\left\{t\left(e^{i(u,c_j)} - 1\right)\mu(A_j)\right\} \\
&= \exp\left\{t\int_A (e^{i(u,f(x))} - 1)\mu(dx)\right\}.
\end{aligned}$$

Now for an arbitrary $f \in L^1(A, \mu_A)$, we can find a sequence of simple functions converging to f in L^1 and hence a subsequence which converges to f almost surely. Passing to the limit along this subsequence in the above yields the required result, via dominated convergence.
(2) and (3) follow from (1) by differentiation. □

It follows from Theorem 6 (2) that a Poisson integral will fail to have a finite mean if $f \notin L^1(A, \mu)$.

For each $f \in L^1(A, \mu_A), t \geq 0$, we define the *compensated Poisson integral* by

$$\int_A f(x)\tilde{N}(t, dx) = \int_A f(x)N(t, dx) - t\int_A f(x)\mu(dx).$$

A straightforward argument shows that $\left(\int_A f(x)\tilde{N}(t, dx), t \geq 0\right)$ is a martingale and we will use this fact extensively in the sequel.

Note that by Theorem 6 (2) and (3), we can easily deduce the following two important facts:

$$\begin{aligned} & \mathbb{E}\left(\exp\left\{i\left(u, \int_A f(x)\tilde{N}(t, dx)\right)\right\}\right) \\ = \; & \exp\left\{t\int_{\mathbb{R}^d}(e^{i(u,x)} - 1 - i(u, x))\mu_{f,A}(dx)\right\}, \end{aligned} \tag{0.3}$$

for each $u \in \mathbb{R}^d$, and for $f \in L^2(A, \mu_A)$,

$$\mathbb{E}\left(\left|\int_A f(x)\tilde{N}(t, dx)\right|^2\right) = t\int_A |f(x)|^2\mu(dx). \tag{0.4}$$

Processes of Finite Variation

We begin by introducing a useful class of functions. Let $\mathcal{P} = \{a = t_1 < t_2 < \cdots < t_n < t_{n+1} = b\}$ be a partition of the interval $[a, b]$ in $\mathbb{R}$, and define its mesh to be $\delta = \max_{1 \leq i \leq n} |t_{i+1} - t_i|$. We define the *variation* $\text{Var}_{\mathcal{P}}(g)$ of a càdlàg mapping $g : [a, b] \to \mathbb{R}^d$ over the partition $\mathcal{P}$ by the prescription

$$\text{Var}_{\mathcal{P}}(g) = \sum_{i=1}^{n} |g(t_{i+1}) - g(t_i)|.$$

If $V(g) = \sup_{\mathcal{P}} \mathrm{Var}_{\mathcal{P}}(g) < \infty$, we say that g has *finite variation on* $[a, b]$. If g is defined on the whole of $\mathbb{R}$ (or $\mathbb{R}^+$), it is said to have *finite variation* if it has finite variation on each compact interval.
It is a trivial observation that every non-decreasing g is of finite variation. Conversely if g is of finite variation, then it can always be written as the difference of two non-decreasing functions - to see this, just write $g = \frac{V(g)+g}{2} - \frac{V(g)-g}{2}$, where $V(g)(t)$ is the variation of g on $[a, t]$.

Functions of finite variation are important in integration, for suppose that we are given a function g which we are proposing as an integrator, then as a minimum we will want to be able to define the Stieltjes integral $\int_I f dg$, for all continuous functions f (where I is some finite interval). In fact a necessary and sufficient condition for obtaining such an integral as a limit of Riemann sums is that g has finite variation.
A stochastic process $(X(t), t \geq 0)$ is of *finite variation* if the paths $(X(t)(\omega), t \geq 0)$ are of finite variation for almost all $\omega \in \Omega$.

The following is an important example for us.
Example *Poisson Integrals*

Let N be a Poisson random measure with intensity measure μ and let $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be Borel measurable. For A bounded below, let $Y = (Y(t), t \geq 0)$ be given by $Y(t) = \int_A f(x)N(t, dx)$, then Y is of finite variation on $[0, t]$ for each $t \geq 0$. To see this, we observe that for all partitions $\mathcal{P}$ of $[0, t]$, we have

$$\mathrm{Var}_{\mathcal{P}}(Y) \leq \sum_{0 \leq s \leq t} |f(\Delta X(s))| \mathbf{1}_A(\Delta X(s)) < \infty \ \text{a.s.} \qquad (0.5)$$

where $X(t) = \int_A x N(t, dx)$, for each $t \geq 0$.

In fact, a necessary and sufficient condition for a Lévy process to be of finite variation is that there is no Brownian part (i.e. $a = 0$ in the Lévy-Khinchine formula) , and $\int_{|x|<1} |x|\nu(dx) < \infty$.

The Lévy-Itô Decomposition

This is the key result of this lecture.
First, note that for A bounded below, for each $t \geq 0$

$$\int_A xN(t, dx) = \sum_{0 \leq u \leq t} \Delta X(u) \mathbf{1}_A(\Delta X(u))$$

is the sum of all the jumps taking values in the set A up to the time t. Since the paths of X are càdlàg, this is clearly a finite random sum. In particular, $\int_{|x| \geq 1} xN(t, dx)$ is the sum of all jumps of size bigger than one. It is a compound Poisson process, has finite variation but may have no finite moments.

On the other hand it can be shown that $X(t) - \int_{|x| \geq 1} xN(t, dx)$ is a Lévy process having finite moments to all orders.
Now lets turn our attention to the small jumps. We study compensated integrals, which we know are martingales. Introduce the notation

$$M(t, A) := \int_A x \tilde{N}(t, dx)$$

for $t \geq 0$ and A bounded below. For each $m \in \mathbb{N}$, let

$$B_m = \left\{ x \in \mathbb{R}^d, \frac{1}{m+1} < |x| \leq \frac{1}{m} \right\}$$

and for each $n \in \mathbb{N}$, let $A_n = \bigcup_{m=1}^{n} B_m$.

Define

$$\int_{|x|<1} x\tilde{N}(t, dx) := L^2 - \lim_{n \to \infty} M(t, A_n),$$

which is a martingale. Moreover, on taking limits in (0.3), we get

$$\mathbb{E}\left(\exp i \left(u, \int_{|x|<1} x\tilde{N}(t, dx) \right) \right) = \exp \left\{ t \int_{|x|<1} (e^{i(u,x)} - 1 - i(u, x)) \mu(dx) \right\}.$$

Consider

$$B_A(t) = X(t) - bt - \int_{|x|<1} x\tilde{N}(t, dx) - \int_{|x| \geq 1} xN(t, dx),$$

where $b = \mathbb{E}\left(X(1) - \int_{|x| \geq 1} xN(1, dx) \right)$. The process B_A is a centred martingale with continuous sample paths. With a little more work, we can show that $\text{Cov}(B_A^i(t) B_A^j(t)) = A^{ij} t$. Using Lévy's characterisation of Brownian motion (see later) we have that B_A is a Brownian motion with covariance a. Hence we have:

Theorem (The Lévy-Itô Decomposition)

If X is a Lévy process, then there exists $b \in \mathbb{R}^d$, a Brownian motion B_a with covariance matrix A in $\mathbb{R}^d$ and an independent Poisson random measure N on $\mathbb{R}^+ \times (\mathbb{R}^d - \{0\})$ such that for each $t \geq 0$,

$$X(t) = bt + B_A(t) + \int_{|x|<1} x\tilde{N}(t, dx) + \int_{|x|\geq 1} xN(t, dx) \qquad (0.6)$$

Note that the three processes in this decomposition are all independent.

An interesting by-product of the Lévy-Itô decomposition is the Lévy-Khintchine formula, which follows easily by independence in the Lévy-Itô decomposition:-

Corollary

If X is a Lévy process, then for each $u \in \mathbb{R}^d, t \geq 0$,

$$\begin{aligned}&\mathbb{E}(e^{i(u,X(t))}) = \\ &\exp\Bigg(t\Bigg[i(b,u) - \frac{1}{2}(u, Au) \\ &+ \int_{\mathbb{R}^d - \{0\}} (e^{i(u,y)} - 1 - i(u,y)\mathbf{1}_B(y))\mu(dy)\Bigg]\Bigg) \qquad (0.7)\end{aligned}$$

so the intensity measure μ is the Lévy measure for X and from now on we write μ as ν.

The process $\int_{|x|<1} x\tilde{N}(t, dx)$ is the *compensated sum of small jumps*. The compensation takes care of the analytic complications in the Lévy-Khintchine formula in a probabilistically pleasing way, since it is an L^2-martingale.
The process $\int_{|x|\geq 1} xN(t, dx)$ describes the "large jumps" - it is a compound Poisson process, but may have no finite moments.

A Lévy process has finite variation iff its Lévy-Itô decomposition takes the form

$$\begin{aligned} X(t) &= \gamma t + \int_{x \neq 0} xN(t, dx) \\ &= \gamma t + \sum_{0 \leq s \leq t} \Delta X(s), \end{aligned}$$

where $\gamma = b - \int_{|x|<1} x\nu(dx)$.

H.Geman, D.Madan and M.Yor have proposed a nice financial interpretation for the jump terms in the Lévy-Itô decomposition:- where the intensity measure is infinite, the stock price manifests "infinite activity" and this is the mathematical signature of the jitter arising from the interaction of pure supply shocks and pure demand shocks. On the other hand, where the intensity measure is finite, we have "finite activity", and this corresponds to sudden shocks that can cause unexpected movements in the market, such as a terrorist atrocity or a major earthquake.

If a pure jump Lévy process (no Brownian part) has finite activity then it has finite variation. The converse is false.

The first three terms on the rhs of (0.6) have finite moments to all orders, so if a Lévy process fails to have a moment, this is due entirely to the "large jumps"/"finite activity" part. In fact:

$$\mathbb{E}(|X(t)|^n) < \infty \text{ for all } t > 0 \text{ if and only if } \int_{|x|\geq 1} |x|^n \nu(dx) < \infty.$$

A Lévy process is a *martingale* iff it is integrable and

$$b + \int_{|x|\geq 1} x\nu(dx) = 0.$$

A square-integrable Lévy process is a martingale iff it is centred and then

$$X(t) = B_A(t) + \int_{\mathbb{R}^d - \{0\}} x\tilde{N}(t, dx).$$

Semimartingales

A stochastic process X is a *semimartingale* if it is an adapted process such that for each $t \geq 0$,

$$X(t) = X(0) + M(t) + C(t),$$

where $M = (M(t), t \geq 0)$ is a local martingale and $C = (C(t), t \geq 0)$ is an adapted process of finite variation. In particular

Every Lévy process is a semimartingale.

To see this, use the Lévy-Itô decomposition to write

$$M(t) = B_a(t) + \int_{|x|<1} x\tilde{N}(t, dx) \text{ - a martingale,}$$

$$C(t) = bt + \int_{|x|\geq 1} xN(t, dx).$$