Review of "Stochastic Analysis: Itô and Malliavin Calculus in Tandem", by H.Matsumoto and S.Taniguchi, Cambridge Studies in Advanced Mathematics **159***, Cambridge University Press (2017), xii + 346 pp*

This book (which is a translation and adaptation of a work by the authors, first published in Japanese as *Kakuritsu Kaiseki* (Bafukan, 2013)) aims to give an introduction to both stochastic analysis and Malliavin calculus, from the point of view of classical path–continuous processes, i.e. Brownian motion, (local) martingales and continuous semimartingales. The material covered is, to a large part standard, as it must be to properly develop the subject from scratch, but there are also many interesting topics, or variations on themes, that are particular to this volume and which help to make it a highly valuable resource. I will emphasise some of these in the chapter–by–chapter survey below. The book is very nicely written, flows well from topic to topic, and is replete with very clearly argued proofs.

Chapter 1 provides essential background on a range of important topics such as Wiener measure, discrete and continuous martingales, stopping times and optional stopping, quadratic variation and the Doob–Meyer decomposition, Brownian motion and the Cameron–Martin formula. It is good to see Schilder's theorem here, which points the way to the study of large deviations. The chapter closes with some discussion of the formal analogy between Wiener integrals and Feynman path integrals, a topic which will recur later in the book. In Chapter 2 we have a full development of stochastic integration with respect to a driving continuous martingale. Itô's formula is proved, used to derive the Burkholder–Davis—Gundy inequality, and to establish Lévy's martingale characterisation of Brownian motion. Next local time is introduced, and applied firstly to extend Itô's formula to convex functions (the Itô–Tanaka formula), and secondly to investigate reflecting Brownian motion, and to characterise this process via the Skorohod equation. The chapter closes with a succinct account of conformal martingales, demonstrating that these can all be realised as time changes of complex Brownian motion.

In Chapter 3 (entitled "Brownian motion and the Laplacian"), the Markov property is introduced, Brownian motion is shown to be a strong Markov process, and this is applied to derive Lévy's formula for the joint density of the position and running maximum of one–dimensional Brownian motion. Next Brownian motion is shown to be recurrent in $\mathbb{R}^d$ for $d \leq 2$, and transient if $d \geq 3$. Probablistic solutions are presented for the heat equation, and for its non–homogeneous extension. This leads naturally to the Feynman–Kac formula. The chapter concludes with a very nice account of the probabilistic solution to the Dirichlet problem on an open set in $\mathbb{R}^d$. Chapter 4 is devoted to stochastic differential equations (SDEs). Existence and uniqueness is proved under the usual Lipschitz assumptions; but alternative conditions for existence (e.g. continuity of coeffcients), and pathwise uniqueness (e.g. Yamada's conditions) are also considered. Next Girsanov's theorem is established, and used to study transformation of drift. A detailed study is made of one–dimensional diffusions; this includes an account of Feller's classification of boundaries, and Khasminskii's condition for non–explosion. Finally conditions are established for the solution to a SDE to give rise to a stochastic flow of diffeomorphisms.

Malliavin calculus is the subject of Chapter 5. This is quite technically demanding. The authors introduce the gradient and divergence operators, and the associated family of Sobolev spaces. Continuity of these and of the Ornstein–Uhlenbeck semigroup, are discussed in the Sobolve space context. The Clark–Ocone and integration by parts formulae are established. The theory is then applied to obtain non–degeneracy of the solutions of SDEs, now viewed as Wiener functionals. A nice application is presented to derive the heat kernel for a Schrödinger operator with a magnetic field, using Donsker's delta function. The integration by parts formula is standard in Malliavin calculus; the change of variables formula is less well–known, and this is presented next, making use of regularised infinite–dimensional determinants. The authors then give an account of quadratic forms on Wiener space, which are defined using the square of the divergence operator. Examples that can be considered from this perspective include the harmonic oscillator Hamiltonian of quantum mechanics, and Lévy's stochastic area. The chapter closes with a brief introduction to rough paths from the point of view of Malliavin calculus applications. The relatively short Chapter 6 gives a standard introductory account of the Balck–Scholes theory of option pricing, with particular emphasis on Malliavin calculus as a tool for computation of the Greeks.

The final Chapter 7 picks up the path integral theme from Chapter 1. In a sense the transition from the Feynman path integral to the Wiener integral requires us to replace real with imaginary time. The authors demonstrate this in the example of the Schrödinger operator with a magnetic field, by showing the analogy between a classical result of von Vleck for the propagator in the fully quantised version, and the heat kernel formula they derived in Chapter 5. Using some of the tools developed here, they next obtain a probabilistic representation of soliton solutions of the Korteweg–de–Vries equation, using Donsker delta functions of components of a two–dimensional Ornstein–Uhlenbeck process. Applying the Feynman–Kac formula, they also investigate the asymptotic distribution of eigenvalues of Schrödinger operators with continuous potentials that are bounded below. The final topic is a sketch of a probabilistic proof of Selberg's trace formula, utilising Brownian motion on hyperbolic space.

The approach of the book is quite sophisticated, and I doubt if it would be accessible to many undergraduates, or MSc students studying on "finance and stochastics" type courses; but it would be highly suitable for graduate students, especially those who have already met some of the ideas in a less–than–fully–rigorous context, and now want to make a deeper study of the subject. I unreservedly recommend this volume to all researchers in stochastic analysis; even the experts can surely learn a lot from it.