On The Infinitesimal Generators of Ornstein-Uhlenbeck Processes with Jumps in Hilbert Space

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Abstract

We study Hilbert space valued Ornstein-Uhlenbeck processes $(Y(t), t \geq 0)$ which arise as weak solutions of stochastic differential equations of the type $dY = JY + CdX(t)$ where J generates a C_0 semigroup in the Hilbert space H, C is a bounded operator and $(X(t), t \geq 0)$ is an H-valued Lévy process. The associated Markov semigroup is of generalised Mehler type. We discuss an analogue of the Feller property for this semigroup and explicitly compute the action of its generator on a suitable space of twice-differentiable functions. We also compare the properties of the semigroup and its generator with respect to the mixed topology and the topology of uniform convergence on compacta.

Keywords: H-valued Lévy process, Ornstein-Uhlenbeck process, generalised Mehler semigroup, auxiliary semigroup, operator-selfdecomposability, quasi-locally equicontinuous semigroup, pseudo-Feller property, mixed topology, cylinder function, Kolmogorov-Lévy operator.

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1 Introduction

The stochastic process called "Brownian motion" is not a good mathematical model for the physical phenomenon of the Brownian movement of e.g. pollen grains immersed in a fluid, as it neglects the viscosity of the medium. To overcome this objection, Ornstein and Uhlenbeck introduced the process which bears their names [27]. It is the unique solution to the Langevin equation:-

$$dy(t) = -\lambda y(t)dt + dB(t), \tag{1.1}$$

where $B = (B(t), t \geq 0)$ is a one-dimensional standard Brownian motion and $\lambda > 0$ measures the viscosity. Interpreting this as a stochastic differential equation, the solution is given in terms of (Wiener) stochastic integrals by:

$$y(t) = e^{-\lambda t}y_0 + \int_0^t e^{-\lambda(t-s)}dB(s). \tag{1.2}$$

There are two ways in which we can usefully generalise this equation. In the first place, we can work in finite or infinite dimensions and replace λ by a suitable matrix or linear operator J. If we are working in a Banach space, it is then natural to ask that J generates a strongly continuous semigroup $S(t) = e^{tJ}$ so that the integral corresponding to that in (1.2) can be defined as a stochastic convolution $\int_0^t S(t-s)dB(s)$ with respect to a Banach-space valued Brownian motion (see e.g.[8]). Secondly we can replace Brownian motion by a more general noise and a natural candidate here is a process with stationary and independent increments, i.e. a Lévy process.

In this paper, we will simultaneously consider both of the generalisations discussed above. We will work with Lévy processes $X = (X(t), t \geq 0)$ taking values in a real separable Hilbert space H. Ornstein-Uhlenbeck processes of this type were first studied by Chojnowska-Michalik [6] and more recently by the author [3]. They are probabilistically interesting for a number of reasons e.g.

- They are solutions of stochastic evolution equations which give an alternative approach to solving stochastic partial differential equations driven by space-time Lévy noise (see section 4.2 in [3]).
- Conditions are known for the existence of invariant measures and these are intimately related to the embedding of operator self-decomposable random variables into the process (see section 5 in [3]).
- The fact that invariant measures can exist suggests that Ornstein-Uhlenbeck processes are better candidates for infinite dimensional "reference processes" than Lévy processes themselves.

The solution of such an equation enjoys the Markov property and so gives rise to a semigroup $(\mathcal{T}_t, t \geq 0)$ on the space of bounded Borel functions on H, taking the form

$$(\mathcal{T}_t f)(x) = \int_H f(S(t)x + y)\rho_t(dy). \tag{1.3}$$

Semigroups of this type are called *generalised Mehler semigroups* and they have been studied extensively by analysts. If $\dim(H) = n < \infty$ and X is a Brownian motion with covariance matrix Q, we have the following explicit formula due to Kolmogorov [18]:

$$(\mathcal{T}_t f)(x) = \frac{1}{(2\pi)^{\frac{n}{2}}(\det(Q(t)))^{\frac{1}{2}}} \int_{\mathbb{R}^n} e^{-\frac{1}{2}\langle Q(t)y, y\rangle} f(S(t)x + y)dy,$$

where each $Q(t) = \int_0^t S(r)QS(r)^* dr$.

Exploring the detailed structure of such "Gaussian Mehler semigroups" is still the subject of intense activity, see e.g. [9], [7], [25] and references therein. Remaining in the finite dimensional setting, but taking X to be a more general Lévy process, the corresponding semigroups have been studied in [32] from a probabilistic viewpoint and [30] from an analytic perspective.

A series of recent papers by M.Röckner and his collaborators have investigated the action of semigroups of the type (1.3) on the space $C_b(H)$ of bounded continuous functions over a real separable Hilbert space ([4], [13], [23], [24], [31]). They have shown that a necessary and sufficient condition for the operators $(\mathcal{T}_t, t \geq 0)$ to enjoy the semigroup property is that $(\rho_t, t \geq 0)$ is a $S(t)$-convolution semigroup (see also [33]), i.e. for all $r, t \geq 0$,

$$\rho_{r+t} = \rho_r * (\rho_t \circ S(r)^{-1}),$$

where $*$ is convolution of measures. Equivalently (under some regularity conditions), in terms of characteristic functions we have for each $x \in H$,

$$\widehat{\rho_t}(x) = \exp\left\{-\int_0^t \eta(S(r)^* x)dr\right\}, \tag{1.4}$$

where $\widehat{\cdot}$ is the Fourier transform and $\eta : H \to \mathbb{C}$ is a negative-definite function. Such semigroups can always be associated (as their transition semigroups) to generalised Ornstein-Uhlenbeck processes driven by H-valued Lévy processes which are constructed via Kolmogorov's existence theorem, however these processes may take their values in a larger Hilbert space in which H is embedded via a Hilbert-Schmidt mapping [4], [13]. Applications of generalized Mehler semigroups to measure-valued catalytic branching processes have been developed in [11]. In the case where an invariant measure τ exists, the generator of the semigroup (now acting on $L^p(H, \tau)$) can be constructed explicitly as an infinite dimensional pseudo-differential operator ([23]).

The aims of the current paper are twofold. Firstly we explore the path from generalised Ornstein-Uhlenbeck processes to Mehler semigroups, when the former are defined via stochastic calculus. In particular we then see that (1.4) arises naturally through the distribution of certain stochastic integrals. We also show that certain identities obtained in ([13]) have a natural probabilistic interpretation using the language of operator self-decomposability. We establish this is section 2 of the paper, which also contains a number of results about Wiener-Lévy stochastic integrals (i.e. those in which the integrand is a sure function) which may be of independent interest. The main business of this paper is found in sections 4 and 5 where we investigate the properties of the infinitesimal generator.

It is well-known that (even in finite dimensions) the semigroup (1.3) is not, in general, strongly continuous on $C_b(H)$ with the usual topology of uniform convergence. When $\dim(H) < \infty$, this problem is overcome by working in the space $C_\infty(H)$ of functions which vanish at infinity (i.e. outside a compact set) and establishing the Feller property. An infinite dimensional analogue of this approach is developed for strong solutions of diffusion type sdes with Lipschitz coefficients in [22] using, in place of $C_\infty(H)$, the space $UC_0(H)$ of uniformly continuous functions which vanish at infinity (outside a bounded set), however we cannot apply this theory to our context (even allowing a natural extension to take care of jumps) because the operator J may be unbounded. Indeed, we conjecture that such a Feller property holds for generalised Mehler semigroups if and only if each $S(t)$ is bounded (c.f. theorem 2.4 in [34]). In section 4, we briefly investigate a "pseudo-Feller property" for Mehler semigroups which requires them to preserve the space $C_0(H)$ of continuous functions which vanish at infinity (outside a bounded set), however even this weaker notion seems to be of limited value.

When a suitable topology is imposed with respect to which the semigroup is strongly continuous (see below), the infinitesimal generator may be defined and is a linear operator on $C_b(H)$. We give an explicit representation of this generator as an integro-differential operator acting on a space of twice-differentiable functions, thus generalising the finite dimensional case given in [32], by making explicit use of Itô's formula. The standard approach to obtaining generators from stochastic differential equations has been generalised to infinite dimensions in [22], for the case of diffusion processes. However, this method requires that the equation has a strong solution. Although strong solutions exist in our context, as pointed out above, they may not live on H. In order to overcome this problem, we apply Itô's formula directly to a certain semimartingale whose distribution is that of the mild solution to the Langevin equation.

Note that the infinite dimensional case requires more structure, in that the operator J plays a role in the construction of the relevant function space. The pseudo-differential operator representation for the generator is obtained as a simple consequence of our result. We establish this directly in a space of continuous functions in contrast to [23] where it was necessary to assume the existence of an invariant measure τ and work in $L^p(H, \tau)(p \geq 1)$. Furthermore, we do not require J^* to have an orthonormal basis of eigenvectors, as in [23].

The problem of lack of strong continuity can be overcome by working in a weaker topology than the usual one induced by the norm, and this has been carried out by a number of authors in different ways, for example the theory of "weakly continuous semigroups" is developed in [5], the notion of π-convergence is used in [29], bi-continuous semigroups are introduced in [20, 21] and the mixed topology is utilised in [15], [16]. We use two topologies in this paper. We first consider the topology of uniform convergence on compacta. The main advantage is that this is easy to use, a disadvantage is that we cannot make use of known results about locally equicontinuous semigroups in locally convex spaces. However there is a related notion of "quasi-locally equicontinuous semigroup", which we introduce in section 3, and which uses the interaction of this topology with the norm topology. Mehler semigroups fit nicely into this class. The infinitesimal generator is densely defined but a distinct disadvantage from an analytic viewpoint is that it seems to be a difficult problem to establish whether or not it is closed, or even closable in this topology.

In the last part of the paper, we consider the mixed topology, i.e. the finest locally convex topology that agrees on norm bounded sets with the topology of uniform convergence on compacts. This has previously been applied to study Mehler semigroups associated with Banach-space valued Ornstein-Uhlenbeck processes driven by Brownian motion in [16] (see also [15]). In this case all of the results established for the topology of uniform convergence on compacta continue to hold, however in addition the generator is closed and (at least when large jumps are well-behaved) has a convenient invariant core of cylinder functions. Consequently we are able to conclude that the mixed topology seems more promising for future work on Mehler semigroups.

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Notation. $\mathbb{R}^+ = [0, \infty)$. If T is a topological space, then $\mathcal{B}(T)$ denotes its Borel σ-algebra. If H is a real separable Hilbert space, $B_b(H)$ is the space of bounded Borel measurable real-valued functions on H and $\mathcal{L}(H)$ is the $*$-algebra of all bounded linear operators on H. I is the identity operator in $\mathcal{L}(H)$. We use B to denote the open unit ball centred on the origin in H and $\hat{B} := B - \{0\}$. The domain of a linear operator T acting in H is denoted as $\text{Dom}(T)$ and its range is $\text{Ran}(T)$. Df denotes the Fréchet derivative of a differentiable H-valued function f defined on H. The Sazonov topology on H is that generated by the family of seminorms $x \rightarrow ||Tx||$, where T ranges over all Hilbert-Schmidt operators in H. If B_1 and B_2 are separable Banach spaces with norms $|| \cdot ||_1$ and $|| \cdot ||_2$, respectively, we say that a mapping $f : B_1 \rightarrow B_2$ *vanishes at infinity* if given any $\epsilon > 0$ there exists a bounded set K in B_1 such that $||f(x)||_2 < \epsilon$ whenever $x \in B_1 - K$.

2 From Lévy Driven Ornstein Uhlenbeck Processes to Generalised Mehler Semigroups

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), P)$ be a stochastic base wherein the filtration $(\mathcal{F}_t, t \geq 0)$ satisfies the usual hypotheses of completeness and right continuity. Let H be a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and associated norm $|| \cdot ||$. Throughout this article, unless contra-indicated, all random variables and processes are understood to be H-valued.

Let $X = (X(t), t \geq 0)$ be an H-valued Lévy process, so X has stationary and independent increments, is stochastically continuous and $X(0) = 0$ (a.s.). In particular it follows that each $X(t)$ is infinitely divisible, so there exists a Sazonov continuous, hermitian, negative definite function $\eta : H \rightarrow \mathbb{C}$ satisfying $\eta(0) = 0$, for which

$$\mathbb{E}(e^{i\langle u, X(t) \rangle}) = e^{-t\eta(u)},$$

for all $t \geq 0, u \in H$. The celebrated Lévy-Khintchine formula asserts that η must be of the form

$$\begin{aligned} \eta(u) &= -i\langle b, u \rangle + \frac{1}{2}\langle u, Qu \rangle \\ &+ \int_{H - \{0\}} [1 - e^{i\langle y, u \rangle} + i\langle y, u \rangle 1_{\hat{B}}(y)]\nu(dy) \end{aligned} \tag{2.5}$$

for each $u \in H$, where $b \in H, Q$ is a positive, self-adjoint, trace class operator on H and ν is a *Lévy measure* on $H-\{0\}$, i.e. $\int_{H-\{0\}}(||y||^2 \wedge 1)\nu(dy) < \infty$. We call the triple (b, Q, ν) the *characteristics* of the process X and the mapping η, the *characteristic exponent* of X. A proof of this result can be found in Chapter 6, section 4 of [28]. The slightly different form given above is discussed e.g. in section 1.2 of [3].

From now on we will always assume that Lévy processes are $\mathcal{F}_t$-adapted and have strongly càdlàg paths. We also strengthen the independent increments requirement on X by assuming that $X(t) - X(s)$ is independent of $\mathcal{F}_s$ for all $0 \leq s < t < \infty$.

The Lévy-Itô decomposition within this context has been established in [12] (see also [1]). It asserts that there exists a Brownian motion $(B_Q(t), t \geq 0)$ with covariance operator Q and an independent Poisson random measure N on $\mathbb{R}^+ \times (H - \{0\})$ with intensity measure $l \otimes \nu$ (where l is Lebesgue measure on $\mathbb{R}^+$) such that

$$X(t) = tb + B_Q(t) + \int_{||x||<1} x\tilde{N}(t, dx) + \int_{||x||\geq 1} xN(t, dx), \qquad (2.6)$$

where $\tilde{N}$ is the compensated Poisson measure, i.e. $\tilde{N}(dt, dx) = N(dt, dx) - dt\nu(dx)$.

The prescription $(T_t f)(x) = \mathbb{E}(f(X(t) + x))$ for each $x \in H, t \geq 0$ defines a Markov semigroup on $B_b(H)$. Now consider the Banach space (equipped with the supremum norm) $UC_0(H)$ of uniformly continuous functions on H which vanish at infinity. Arguing as in section 3.1 of [2], we see that $(T_t, t \geq 0)$ is a Feller semigroup in that for each $t \geq 0, f \in UC_0(H)$,

$$T_t(UC_0(H)) \subseteq UC_0(H) \quad \text{and} \quad \lim_{t \to 0} ||T_t f - f|| = 0.$$

We denote the infinitesimal generator of $(T_t, t \geq 0)$ by $\mathcal{A}_X$.

Let $UC_2(H)$ be the subspace of $UC_0(H)$ comprising all C^2 functions from H to $\mathbb{R}$ for which $Df : H \to H$ and $D^2 f : H \to \mathcal{L}(H)$ are uniformly continuous and vanish at infinity. We may apply Itô's formula (see [26] theorem 27.2) and a standard stochastic calculus argument (see e.g. theorem 6.7.4 in [2]) to deduce that $UC_2(H) \subseteq \text{Dom}(\mathcal{A}_X)$ and for all $f \in UC_2(H), x \in H$,

$$\begin{aligned}(\mathcal{A}_X f)(x) &= \langle (Df)(x), b \rangle + \frac{1}{2}\text{tr}((D^2 f)(x)Q) \\ &+ \int_{H-\{0\}} [f(x+y) - f(x) - \langle (Df)(x), y \rangle 1_{\hat{B}}(y)]\nu(dy). \quad (2.7)\end{aligned}$$

Let F be a measurable function from $\mathbb{R}^+$ to $\mathcal{L}(H)$ which is such that the mapping $t \to ||F(t)||$ is locally square integrable. We may define the integral $\int_0^t F(s)dX(s)$ by the procedure of [3] (see section 4.1 therein) via (2.6) :

$$\begin{aligned}\int_0^t F(s)dX(s) &:= \int_0^t F(s)bds + \int_0^t F(s)dB_Q(s) \\ &+ \int_0^t \int_{||x||<1} F(s)x\tilde{N}(ds,dx) + \int_0^t \int_{||x||\geq 1} F(s)xN(ds,dx).\end{aligned}$$

The first and fourth of these integrals are defined as standard Bochner integrals (indeed, the fourth term is just a random finite sum) while the second and third terms are constructed as stochastic integrals and each of these is in fact an L^2-martingale. Alternative methods of constructing such integrals can be found in [17] (via integration by parts) and [6] (via convergence in probability).

We will require the following result in section 4:-

Lemma 2.1 *If the mapping $t \to ||F(t)||$ is locally bounded, then*

(i) $\int_0^t F(s)dX(s)$ *is stochastically continuous.*

(ii) $\int_0^t F(t-s)dX(s)$ *is stochastically continuous.*

Proof.

(i) We may assume that F is non-zero. For each $t \geq 0$, we write

$$M(t) := \int_0^t F(s)dB_Q(s) + \int_0^t \int_{||x||<1} F(s)x\tilde{N}(ds,dx).$$

By the Itô-type isometry established in section 3.2 of [3], we have

$$\begin{aligned}\mathbb{E}(||M(t)-M(s)||^2) &= \int_s^t \mathrm{tr}(F(r)QF(r)^*)dr \\ &+ \int_s^t \int_{||x||<1} \mathrm{tr}(F(r)T_xF(r)^*)\nu(dx)dr,\end{aligned}$$

for each $0 \leq s < t < \infty$, where for each $x \in H, T_x(\cdot) = \langle x, \cdot \rangle x$. Stochastic continuity of $M(t)$ follows by a standard application of Chebychev's inequality. For the "big jumps" term, we have for all $a > 0$,

$$\begin{aligned}
&P\left(\left|\left|\int_s^t \int_{||x||>1} F(r)xN(dr,dx)\right|\right| > a\right)\\
\leq\ & P\left(\sup_{0\leq r\leq t}||F(r)||.\left|\left|\int_{||x||>1} xN(t,dx) - \int_{||x||>1} xN(s,dx)\right|\right| > a\right)\\
\rightarrow\ & 0 \text{ as } s\rightarrow t
\end{aligned}$$

by stochastic continuity of the Lévy process $\left(\int_{||x||>1} xN(t,dx), t\geq 0\right)$.

(ii) For all $a>0, 0\leq s<t<\infty$,

$$\begin{aligned}
P\left(\left|\left|\int_s^t F(t-r)dX(r)\right|\right| > a\right) &\leq P\left(\left|\left|\int_s^t (F(t-r)-F(r))dX(r)\right|\right| > \frac{a}{2}\right)\\
&+ P\left(\left|\left|\int_s^t F(r)dX(r)\right|\right| > \frac{a}{2}\right)\\
&\leq P\left(\sup_{0\leq r\leq t}||F(r)||.||X(t)-X(s)|| > \frac{a}{4}\right)\\
&+ P\left(\left|\left|\int_s^t F(r)dX(r)\right|\right| > \frac{a}{2}\right),
\end{aligned}$$

and the result follows by (i) and the stochastic continuity of the process X. □

Let $(\mathcal{P}_n, n\in\mathbb{N})$ be a sequence of partitions of $[0,t]$ with each $\mathcal{P}_n$ having mesh δ_n with $\lim_{n\rightarrow\infty}\delta_n = 0$. We write each $\mathcal{P}_n = \{0 = t_1^{(n)} < t_2^{(n)} \cdots < t_{m_n+1}^{(n)} = t\}$.

From the above construction it follows that

$$\int_0^t F(s)dX(s) = \lim_{n\rightarrow\infty}\sum_{j=1}^{m_n} F(t_j^{(n)})(X(t_{j+1}^{(n)}) - X(t_j^{(n)})),$$

where the limit is taken in the sense of convergence in distribution.

Proposition 2.1 *For each $t\geq 0, \int_0^t F(s)dX(s)$ is infinitely divisible and its characteristic exponent is given by*

$$\lambda_{t,F}(u) := \int_0^t \eta(F(s)^* u)ds, \tag{2.8}$$

for each $u\in H$.

Proof. We first note that the integral on the right hand side exists, indeed using a well-known estimate for characteristic exponents (see e.g. [2], p.30) we may assert that there exists $K > 0$ such that for all $s \geq 0, u \in H$,

$$\begin{aligned} ||\eta(F(s)^*u)||^2 &\leq K(1 + ||F(s)^*u||^2) \\ &\leq K(1 + ||F(s)||^2||u||^2). \end{aligned}$$

Let $\mathcal{P} = \{0 = t_1 < t_2 < \cdots < t_{n+1} = t\}$ be a partition of $[0, t]$ and define $Y_{F,\mathcal{P}} = \sum_{j=1}^{n} F(t_j)(X(t_{j+1}) - X(t_j))$. Using independent increments of X and (2.5) we obtain for each $u \in H$,

$$\begin{aligned} \mathbb{E}(e^{i\langle u, Y_{F,\mathcal{P}}\rangle}) &= \mathbb{E}\left(\exp\left\{i\left\langle u, \sum_{j=1}^{n} F(t_j)(X(t_{j+1}) - X(t_j))\right\rangle\right\}\right) \\ &= \prod_{j=1}^{n} \mathbb{E}(\exp\{i\langle F(t_j)^*u, (X(t_{j+1}) - X(t_j))\rangle\}) \\ &= \prod_{j=1}^{n} \exp\{-(t_{j+1} - t_j)\eta(F(t_j)^*u)\} \\ &= \exp\left\{-\sum_{j=1}^{n}(t_{j+1} - t_j)\eta(F(t_j)^*u)\right\} \end{aligned}$$

Now replace $\mathcal{P}$ by $\mathcal{P}_n$ within a suitable sequence of partitions $(\mathcal{P}_n, n \in \mathbb{N})$ and take limits to obtain,

$$\mathbb{E}\left(\exp\left\{i\left\langle u, \int_0^t F(s)dX(s)\right\rangle\right\}\right) = \exp\left\{-\int_0^t \eta(F(s)^*u)ds\right\},$$

To see that the stochastic integral is infinitely divisible, first note that for each $n \in \mathbb{N}, \frac{\eta}{n}$ is Sazonov continuous, hermitian, negative definite and vanishing at zero, hence there exists a càdlàg Lévy process $(X_n(t), t \geq 0)$ such that for each $u \in H, t \geq 0$,

$$\mathbb{E}(e^{i\langle u, X_n(t)\rangle}) = e^{-t\frac{\eta(u)}{n}}.$$

Hence

$$\left[\mathbb{E}\left(\exp\left\{i\left\langle u, \int_0^t F(s)dX(s)\right\rangle\right\}\right)\right]^{\frac{1}{n}} = \mathbb{E}(e^{i\langle u, \int_0^t F(s)dX_n(s)\rangle}),$$

and the result follows. $\square$

Notes 1. The formula (2.8) can also be found in Corollary 1.7 of [6] but the approach there is quite different.

2. By a straightforward extension of the last part of the proof of Proposition 2.1 it can be shown that $(I_F(t), t \geq 0)$, where each $I_F(t) := \int_0^t F(s)dX(s)$, is an infinitely divisible process, i.e. for each $n \in \mathbb{N}$ and $t_1, t_2, \ldots, t_n \in \mathbb{R}^+$, the random vector $(I_F(t_1), I_F(t_2), \ldots, I_F(t_n))$ is infinitely divisible. This also follows from the fact that $(I_F(t), t \geq 0)$ is an additive process, which may be proved by the argument of lemma 4.3.12 in [2].

Corollary 2.1 *For each $t \geq 0$, $\int_0^t F(s)dX(s)$ has characteristics (b_t, Q_t, ν_t), where*

$$b_t := \int_0^t F(s)bds + \int_0^t \int_{H-\{0\}} F(s)x[1_{\hat{B}}(x) - 1_{\hat{B}}(F(s)x)]\nu(dx)ds,$$

$$Q_t := \int_0^t F(s)QF(s)^* ds,$$

$$\nu_t(A) := \int_0^t \nu(F(s)^{-1}A)ds, \ \textit{for each } A \in \mathcal{B}(H - \{0\}).$$

Proof. This is an immediate consequence of Proposition 2.1 and (2.5). $\square$

Note. It follows that $(I_F(t), t \geq 0)$ is a Gaussian process if and only if $\nu \equiv 0$ and $F(\cdot)QF(\cdot)^* \neq 0$ on a subset of $\mathbb{R}^+$ which has strictly positive Lebesgue measure.

For each $t \geq 0$ let μ_t denote the law of $\int_0^t F(s)dX(s)$.

Proposition 2.2 *For each $T > 0$, the set $\{\mu_t, 0 \leq t \leq T\}$ is uniformly tight.*

Proof. This follows from applying Theorem 5.3 in [28] to the result of Corollary 2.1 via the argument of [13], page 19. $\square$

Let $(S(t), t \geq 0)$ be a strongly continuous semigroup in H with infinitesimal generator J and fix $C \in \mathcal{L}(H)$. From now on, for each fixed $t \geq 0$, we let $F(s) = S(t-s)C1_{[0,t]}(s)$. Consider the stochastic differential equation (s.d.e.)

$$dY(t) = JY(t)dt + CdX(t) \tag{2.9}$$

with initial condition $Y(0) = Y_0$ (a.s.) for some given $\mathcal{F}_0$-measurable random variable Y_0. In [6], [3] it was shown that the unique weak solution of this s.d.e. is the *Lévy driven Ornstein-Uhlenbeck process* $Y = (Y(t), t \geq 0)$ where

$$Y(t) = S(t)Y_0 + \int_0^t S(t-s)CdX(s).$$

We call $(S(t), t \geq 0)$ the *auxiliary semigroup* of Y.

It follows from lemma 2.1 (ii) that Y is stochastically continuous, indeed by a standard estimate for C_0-semigroups (see e.g. [10]) we can assert that there exists $M > 1$ and $\beta \in \mathbb{R}$ such that

$$||F(s)|| \leq M||C||e^{\beta(t-s)}, \tag{2.10}$$

for all $0 \leq s \leq t < \infty$, from which the local boundedness requirement follows easily.

Y is a strong Markov process and the associated Markov semigroup acting on $C_b(H)$ is $(\mathcal{T}_t, t \geq 0)$ where for each $f \in C_b(H), x \in H$,

$$\begin{aligned}(\mathcal{T}_t f)(x) &= \mathbb{E}(f(Y(t))|Y_0 = x) \\ &= \int_H f(S(t)x + y)\rho_t(dy), \qquad (2.11)\end{aligned}$$

where ρ_t is the law of $\int_0^t S(t-s)CdX(s)$. Note that ρ_t is infinitely divisible by proposition 2.1.

Semigroups of the type (2.11) are called *generalised Mehler semigroups.* They have been studied systematically in a number of recent papers - see [4], [13],[23], [24]. In fact, through the construction of [13], we can assert that there is a one-to-one correspondence between such semigroups and strong solutions of (2.9), although such solutions may have to be constructed on a larger Hilbert space in which H is embedded via a Hilbert-Schmidt mapping.

In [3], the notion of operator self-decomposability was generalised as follows: a random variable Z is operator self-decomposable with respect to the C_0-semigroup $(S(t), t \geq 0)$ if for each $t \geq 0$ there exists a random variable Z_t, which is independent of Z such that

$$Z \stackrel{d}{=} S(t)Z + Z_t, \tag{2.12}$$

or equivalently

$$p_Z = (p_Z \circ S(t)^{-1}) * p_{Z_t}. \tag{2.13}$$

Let μ be an invariant Borel probability measure for $(\mathcal{T}_t, t \geq 0)$ so that

$$\int_H (\mathcal{T}_t f)(x)\mu(dx) = \int_H f(x)\mu(dx),$$

for all $f \in B_b(H), t \geq 0$. It follows from (2.13) (as established in section 3 of [6] - see also section 3 of [13], but without the explicit connection with (2.12)) that μ is invariant for $(\mathcal{T}_t, t \geq 0)$ if and only if it is operator self-decomposable with respect to $(S(t), t \geq 0)$ with each $Z_t \stackrel{d}{=} \int_0^t S(t-s)CdX(s)$. In fact we have the following:

Theorem 2.1 *Let $Y = (Y(t), t \geq 0)$ be a Lévy driven Ornstein-Uhlenbeck process with auxiliary semigroup $(S(t), t \geq 0)$ and associated generalised Mehler semigroup $(\mathcal{T}_t, t \geq 0)$. The following are equivalent:-*

1. *μ is invariant for $(\mathcal{T}_t, t \geq 0)$.*
2. *$(Y(t), t \geq 0)$ is (strictly) stationary.*
3. *Y_0 is operator self-decomposable with respect to $(S(t), t \geq 0)$ with $Z_t \stackrel{d}{=} \int_0^t S(t-s)CdX(s)$ for each $t \geq 0$.*

Proof. The equivalence of (1) and (2) is a well-known fact about stationary Markov processes (see e.g. Proposition 11.5 in [8]). That of (1) and (3) follows from the discussion above. □

For further discussion of the circle of ideas described in theorem 2.1 (for infinite dimensional H) see section 3 of [13] and section 5 in [3]. Necessary and sufficient conditions for Y to have an invariant probability measure are obtained in [6] (see also theorem 3.1 in [13]). For the finite dimensional case, see section 4.3.5 of [2] and references therein.

3 Quasi-Locally Equicontinuous Semigroups

Let $\mathbb{B}$ be a real Banach space with norm $||\cdot||$. We assume that $\mathbb{B}$ is also equipped with a locally convex sequentially complete topology τ, which is weaker than the norm topology. We fix a generating family of seminorms $\mathcal{P}_{\mathcal{I}} := (p_i, i \in \mathcal{I})$, where $\mathcal{I}$ is some index set.

Let $(V(t), t \geq 0)$ be a semigroup of bounded linear operators on $\mathbb{B}$ which is also τ-strongly continuous. We say that it is *quasi-locally equicontinuous* if for each $T > 0$ and for every $p \in \mathcal{P}_{\mathcal{I}}$, there exists $K_{p,T} > 0$ such that

$$p(V(t)(u)) \leq K_{p,T}||u||, \tag{3.14}$$

for each $u \in \mathbb{B}, 0 \leq t \leq T$. It is said to be *quasi-equicontinuous* if $K_{p,T}$ can be chosen to be independent of T.

We define the τ-infinitesimal generator $\mathcal{M}$ of $(V(t), t \geq 0)$ in the obvious way, so its domain is given by

$$\text{Dom}(\mathcal{M}) = \{f \in \mathbb{B}, \text{there exists } g \in \mathbb{B} \text{ such that } \tau - \lim_{t \to 0} \frac{V_t f - f}{t} = g\},$$

$$\text{and} \quad \mathcal{M}f = g,$$

for all $f \in \text{Dom}(\mathcal{M})$ then defines a linear operator in $\mathbb{B}$.

By sequential completeness of $\mathbb{B}$ and the continuity of $t \to V(t)u$ in $(\mathbb{B}, \tau)$, the Riemann integral $\int_0^t V(s)uds$ exists and defines a vector in $\mathbb{B}$, for each $u \in \mathbb{B}$.

Proposition 3.1 *Let $(V(t), t \geq 0)$ be a quasi-locally equicontinuous semigroup in a real Banach space $\mathbb{B}$.*

(i) $x \in$ Dom$(\mathcal{M})$ and $\mathcal{M}x = y \in \mathbb{B}$ if and only if

$$V(t)x - x = \int_0^t V(s)yds \quad \text{for all } t \geq 0.$$

(ii) Dom$(\mathcal{M})$ is τ-dense in $\mathbb{B}$.

(iii) $\mathcal{M}$ is closed from $(\mathbb{B}, \tau)$ to $(\mathbb{B}, ||\cdot||)$.

Proof. (i) and (ii) are established (under greater generality) in [19], propositions 1.2 and 1.3.

(iii) is a variation on the standard proof of closedness of generators of one parameter semigroups on Banach spaces (see e.g. [10]). We include the argument for completeness.

Let $(\psi_n, n \in \mathbb{N})$ be a sequence of vectors in $\mathbb{B}$ for which $\tau - \lim_{n\to\infty} \psi_n = \psi \in \mathbb{B}$ and $||\cdot|| - \lim_{n\to\infty} \mathcal{M}\psi_n = \phi \in \mathbb{B}$. From (i), we have for each $t \geq 0$,

$$\begin{aligned} V(t)\psi - \psi &= \tau - \lim_{n\to\infty} (V(t)\psi_n - \psi_n) \\ &= \tau - \lim_{n\to\infty} \int_0^t V(s)\mathcal{M}\psi_n ds. \end{aligned}$$

However given any $T > 0$ and any $p \in \mathcal{P}_{\mathcal{I}}$, for all $0 \leq t \leq T$, we have

$$\begin{aligned} \lim_{n\to\infty} p\left(\int_0^t V(s)\mathcal{M}\psi_n ds - \int_0^t V(s)\phi ds\right) &\leq \lim_{n\to\infty} \int_0^t p[V(s)(\mathcal{M}\psi_n - \phi)]ds \\ &\leq TK_{p,T} \lim_{n\to\infty} ||\mathcal{M}\psi_n - \phi|| = 0. \end{aligned}$$

Hence

$$V(t)\psi - \psi = \int_0^t V(s)\phi ds,$$

and so $\phi = \mathcal{M}\psi$ by (i). □

We remark that quasi-locally equicontinuous semigroups are closely related to, but appear to be distinct from bi-continuous semigroups as introduced in [20, 21].

4 The Pseudo-Feller Property and Generators of Generalised Mehler Semigroups

As is pointed out in section 6.3 in [9] (see also [5]), $(\mathcal{T}_t, t \geq 0)$ is not strongly continuous in general on $C_b(H)$, with the usual topology τ_0 of uniform convergence. We will sometimes denote the supremum norm in $C_b(H)$ by $||\cdot||_0$. In this section, we will consider $C_b(H)$ equipped with the topology τ_{uc} of uniform convergence on compacta. If $(f_n, n \in \mathbb{N})$ is a sequence in $C_b(H)$ converging in τ_{uc} to $f \in C_b(H)$, we write $f_n \stackrel{uc}{\to} f$. We also find it convenient to introduce the mixed topology τ_m on $C_b(H)$ which is defined as follows (see [15]). Let $\mathcal{K}$ denote the set of all sequences $(K_n, n \in \mathbb{N})$ of compact subsets of H and let $\mathcal{S}$ denote the set of all positive null sequences. The mixed topology is induced by the family of seminorms $\{\rho_{(a_n),(K_n)}, (a_n, n \in \mathbb{N}) \in \mathcal{S}, (K_n, n \in \mathbb{N}) \in \mathcal{K}\}$ where for each $f \in C_b(H)$,

$$\rho_{(a_n),(K_n)}(f) = \sup_{n\in\mathbb{N}} \sup_{x \in K_n} |a_n f(x)|.$$

τ_m is complete and sequential convergence is characterised as follows: a sequence $(f_n, n \in \mathbb{N})$ in $C_b(H)$ converges in τ_m to $f \in C_b(H)$ if

M(i) $f_n \stackrel{uc}{\to} f$, as $n \to \infty$,

M(ii) $\sup_{n\in\mathbb{N}} ||f_n||_0 < \infty$.

See [16], [15] for full proofs and further development of these ideas.

We now return to the study of the Mehler semigroup $(\mathcal{T}_t, t \geq 0)$.

Theorem 4.1 $(\mathcal{T}_t, t \geq 0)$ *is uc-strongly continuous.*

Proof. (c.f. Proposition 6.2 in [5], theorem 4.2 in [13] and theorem 6.2 in [16]).

Let $f \in C_b(H)$. By Proposition 2.2, $\{\rho_t, t \in [0,1]\}$ is uniformly tight, hence given an arbitrary $\epsilon > 0$, there exists a compact set L in H such that

$$\rho_t(L) \geq 1 - \frac{\epsilon}{8||f||} \quad \text{for all} \quad t \in [0,1].$$

Fix a compact $K \subset H$. As in the proof of theorem 6.2 in [16] we can use uniform continuity of f on compacta and strong continuity of $(S(t), t \geq 0)$ to argue that there exists $t_0 \in [0,1]$ such that $0 \leq t < t_0 \Rightarrow$

$$\sup_{x \in K} \sup_{y \in L} |f(S(t)x + y) - f(x+y)| < \frac{\epsilon}{4}.$$

Now write

$$(\mathcal{T}_t f)(x) - f(x) = I_1(f,t,x) + I_2(f,t,x),$$

where

$$I_1(f,t,x) := \int_H [f(S(t)x + y) - f(x+y)]\rho_t(dy)$$
$$\text{and} \quad I_2(f,t,x) := \int_H [f(x+y) - f(x)]\rho_t(dy).$$

Now for $0 \leq t < t_0$,

$$\begin{aligned} \sup_{x \in K} |I_1(f,t,x)| &\leq \sup_{x \in K} \int_H |f(S(t)x+y) - f(x+y)|\rho_t(dy) \\ &\leq \sup_{x \in K} \sup_{y \in L} |f(S(t)x+y) - f(x+y)| + 2||f||\rho_t(L^c) \\ &< \frac{\epsilon}{2} \end{aligned}$$

Using uniform continuity of f on compacta and stochastic continuity of Y, we can argue as in the proof of theorem 3.1.9 in [2] that $\sup_{x \in K} |I_2(f,t,x)| < \frac{\epsilon}{2}$, for sufficiently small t. Hence $\mathcal{T}_t f \stackrel{uc}{\to} f$ as $t \to 0$ and the result follows. □

Taking the locally convex topology in the previous section to be τ_{uc} with $\mathcal{I}$ being the set of all compact subsets of H and each $p_K(f) = \sup_{x \in K} |f(x)|$, for $K \in \mathcal{I}, f \in C_b(H)$ we immediately deduce the following.

Corollary 4.1 $(\mathcal{T}_t, t \geq 0)$ *is quasi-equicontinuous.*

Let $C_0(H)$ be the space of continuous functions on H which vanish at infinity (i.e. outside a bounded set). We say that a conservative, positivity preserving contraction semigroup $(V(t), t \geq 0)$ is *pseudo-Feller* if it is uc-strongly continuous and

$$V(t)(C_0(H)) \subseteq C_0(H),$$

for all $t \geq 0$.

Suppose that $(\mathcal{T}_t, t \geq 0)$ is a Mehler semigroup with auxiliary semigroup $(S(t), t \geq 0)$. We make the assumption that

$$||S(t)x|| \to \infty \quad \text{whenever} \quad ||x|| \to \infty, \tag{4.15}$$

for all $t \geq 0$. It is then clear that $(\mathcal{T}_t, t \geq 0)$ is pseudo-Feller. (4.15) clearly holds when each $S(t)$ is an isometry. We also see easily that this condition holds when J is self-adjoint and bounded, indeed let $\sigma(J) = [-\alpha, -\beta]$ (say) be the spectrum of J and for each $-\alpha \leq \lambda \leq -\beta$, let E_λ be the corresponding spectral projection associated to the set $[-\alpha, -\lambda]$. Then for all $t \geq 0$,

$$\begin{aligned} ||S(t)x||^2 &= \int_{-\alpha}^{-\beta} e^{2ty}||E(dy)x||^2 \\ &\geq e^{-2\alpha t}||x||^2 \\ &\to \infty \text{ as } ||x|| \to \infty, \end{aligned}$$

and we are done.

Note. In fact if J is bounded, $(\mathcal{T}_t, t \geq 0)$ is pseudo-Feller by theorem 5.19 in [22]. In this case, the semigroup also satisfies the usual Feller property, in that it is strongly continuous (with respect to the supremum norm) on $UC_0(H)$.

The following counter-example, which was suggested to the author by Ben Goldys, suggests that (4.15) is of limited value. Let $(x_n, n \in \mathbb{N})$ be a sequence in H which converges weakly to zero, but each $||x_n|| = 1$. Suppose that the semigroup $(S(t), t \geq 0)$ comprises compact operators, so that each $(S(t)x_n, n \in \mathbb{N})$ converges strongly to zero. For each $t \geq 0$, define a sequence $y_n(t) = \dfrac{x_n}{||S(t)x_n||}$. Then as $n \to \infty$, $||y_n(t)|| \to \infty$, however $||S(t)y_n(t)|| \to 1$.

We will not require the condition 4.15 elsewhere in this paper.

We denote the infinitesimal generator of $(\mathcal{T}_t, t \geq 0)$ by $\mathcal{A}_Y$. In order to compute its action we introduce the following space.

We define $C_J^2(H)$ to be the linear subspace of $C_b(H)$ comprising those C^2 functions f whose first and second derivatives are uniformly bounded and uniformly continuous on bounded subsets of H and for which

$$\mathrm{Ran}(Df) \subseteq \mathrm{Dom}(J^*) \text{ and the mapping } x \to \langle x, J^*(Df)(x)\rangle \in C_b(H).$$

It is also convenient at this stage to introduce the space $\mathcal{F}C_J^2(H)$ of cylinder functions which are defined as follows: $F_f \in \mathcal{F}C_J^2(H)$ if $F_f \in C_J^2(H)$ and there exists $n \in \mathbb{N}, x_1, \ldots, x_n \in \mathrm{Dom}(J^*)$ and $f \in C_b^2(\mathbb{R}^n)$ such that

$$F_f(x) = f(\langle x_1, x\rangle, \ldots, \langle x_n, x\rangle), \tag{4.16}$$

for each $x \in H$. If $F_f \in \mathcal{F}C_J^2(H)$, then it is easy to see that

$$(DF_f)(x) = \sum_{j=1}^{n} (\partial_j f)(\langle x_1, x\rangle, \ldots, \langle x_n, x\rangle)x_j.$$

To see that $\mathcal{F}C_J^2(H)$ is non-empty observe that it contains the set $\mathcal{F}C_{c,J}^\infty(H)$ of cylinder functions wherein each f is infinitely differentiable with compact support. It is shown in lemma 2.6 of [15] that $\mathcal{F}C_{c,J}^\infty(H)$ is τ_m-dense in $C_b(H)$. It follows that it is τ_{uc}-dense in $C_b(H)$.

If $C \in \mathcal{L}(H)$ and X is a Lévy process with characteristics (b, Q, ν), we denote as X_C the process $(CX(t), t \geq 0)$. It is easily verified that X_C is another Lévy process with characteristics
$\left(Cb + \int_{H-\{0\}} Cy[1_{\hat{B}}(y) - 1_{\hat{B}}(Cy)]\nu(dy), CQC^*, \nu_C\right)$, where $\nu_C = \nu \circ C^{-1}$.

We introduce the linear *Kolmogorov-Lévy* operator $\mathcal{L} : C_J^2(H) \to C_b(H)$ defined by

$$(\mathcal{L}f)(x) := (\mathcal{A}_{X_C} f)(x) + \langle J^*(Df)(x), x\rangle, \tag{4.17}$$

for each $x \in H$ where $\mathcal{A}_{X_C}$ is of the form (2.7) with X replaced by X_C therein.

Theorem 4.2 *$C_J^2(H) \subseteq Dom(\mathcal{A}_Y)$ and*

$$\mathcal{A}_Y f = \mathcal{L}f,$$

for all $f \in C_J^2(H)$.

Proof. By the above discussion we see that we may without loss of generality take $C = I$.

First assume that $x \in \mathrm{Dom}(J)$ then (conditioning on the event $Y_0 = x$), we have for each $t \geq 0$,

$$
\begin{aligned}
Y(t) &= S(t)x + \int_0^t S(t-s)dX(s) \\
&= x + \int_0^t S(t-s)Jxds + \int_0^t S(t-s)dX(s) \\
&= x + \int_0^t S(t-s)d\widehat{X}(s),
\end{aligned}
$$

where $(\widehat{X}(t), t \geq 0)$ is a Lévy process with characteristics $(b + Jx, Q, \nu)$.

As Y is not a semimartingale, we cannot directly apply Itô's formula to it. However $Y_1 = (Y_1(t), t \geq 0)$ is a semimartingale where for each $t \geq 0, x \in H$,

$$
Y_1(t) = x + \int_0^t S(r)d\widehat{X}(r).
$$

Furthermore, by stationary increments of $\widehat{X}$, each $Y_1(t)$ has the same distribution of $Y(t)$ and hence both Markov processes have the same transition semigroup.

We can now apply Itô's formula (see [26], theorem 27.2) to obtain for each $f \in C_J^2(H)$,

$$
\begin{aligned}
f(Y_1(t)) &= f(x) + \text{a martingale} + \int_0^t \int_{||y||>1} f(Y_1(u-) + S(u)y) - f(Y_1(u-))N(du, dy) \\
&+ \int_0^t \Big\{ \langle (Df)(Y_1(u-)), S(u)b + Jx \rangle + \frac{1}{2}\text{tr}((D^2 f)(Y_1(u-))S(u)QS(u)^*) \\
&+ \int_{||y|| \leq 1} [f(Y_1(u-) + S(u)y) - f(Y_1(u-))] \\
&- \langle (Df)(Y_1(u-), S(u)y \rangle \nu(dy) \} \, du.
\end{aligned}
$$

Now

$$
\begin{aligned}
(\mathcal{T}_t f)(x) - f(x) &= \mathbb{E}(f(Y_1(t))|Y_0 = x) - f(x) \\
&= \int_0^t \{ \mathcal{T}_u(\langle (Df)(x), (Jx + S(u)b) \rangle) \\
&+ \frac{1}{2}\mathcal{T}_u(\text{tr}[(D^2 f)(x)S(u)QS(u)^*]) \\
&+ \int_{H-\{0\}} [\mathcal{T}_u f(x + S(u)y) - \mathcal{T}_u f(x) \\
&- \mathcal{T}_u(\langle (Df)(x), S(u)y \rangle) 1_{\hat{B}}(y)] \nu(dy) \} \, du.
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^t \{\mathcal{T}_u[(\langle J^*(Df)(x), x\rangle + \langle (Df)(x), S(u)b\rangle) \\
&+ \frac{1}{2}\mathcal{T}_u(\mathrm{tr}[(D^2 f)(x)S(u)QS(u)^*]) \\
&+ \int_{H-\{0\}} [\mathcal{T}_u f(x+S(u)y) - \mathcal{T}_u f(x) \\
&- \mathcal{T}_u(\langle (Df)(x), S(u)y\rangle)1_{\hat{B}}(y)]\nu(dy)\}\, du,
\end{aligned}
$$

which extends to all $x \in H$, by continuity.

Let $K \subset H$ be compact. In the following we will employ the notation $||g||_K := \sup_{x\in K} |g(x)|$.

We observe that

$$
\sup_{x\in K}\left|\frac{1}{t}(\mathcal{T}_t f(x) - f(x)) - (\mathcal{L}f)(x)\right| \le I_t^{(1)}(f) + I_t^{(2)}(f) + I_t^{(3)}(f) + I_t^{(4)}(f),
$$ where

$$
\begin{aligned}
I_t^{(1)}(f) &:= \frac{1}{t}\sup_{x\in K}\left|\int_0^t [\mathcal{T}_u[(\langle J^*(Df)(x), x\rangle + \langle (Df)(x), S(u)b\rangle)]du \right.\\
&- \left.\langle J^*(Df)(x), x\rangle - \langle (Df)(x), b\rangle\right|,
\end{aligned}
$$

$$
I_t^{(2)}(f) := \frac{1}{2t}\sup_{x\in K}\left|\int_0^t \mathcal{T}_u[\mathrm{tr}((D^2 f)(x)S(u)QS(u)^*)]dr - \mathrm{tr}((D^2 f)(x)Q)\right|,
$$

$$
\begin{aligned}
I_t^{(3)}(f) &:= \frac{1}{t}\sup_{x\in K}\left|\int_0^t \int_{||y||\ge 1} [\mathcal{T}_u f(x+S(u)y) - \mathcal{T}_u f(x)]\nu(dy)du \right.\\
&- \left.\int_{||y||\ge 1} [f(x+y) - f(x)]\nu(dy)\right|,
\end{aligned}
$$

$$
\begin{aligned}
I_t^{(4)}(f) &:= \frac{1}{t}\sup_{x\in K}\left|\int_0^t \int_{||y||<1} [\mathcal{T}_u f(x+S(u)y) - \mathcal{T}_u f(x) \right.\\
&- \mathcal{T}_u(\langle (Df)(x), S(u)y\rangle)]\nu(dy)du \\
&- \left.\int_{||y||<1} [f(x+y) - f(x) - \langle (Df)(x), y\rangle]\nu(dy)\right|.
\end{aligned}
$$

We have

$$
I_t^{(1)}(f) \le \frac{1}{t}\int_0^t \sup_{x\in K} |(\mathcal{T}_u - I)(\langle J^*(Df)(x), x\rangle)|du
$$

$$
\begin{aligned}
&+ \quad \frac{1}{t}\int_0^t \sup_{x\in K}|\mathcal{T}_u(\langle (Df)(x), S(u)b\rangle\rangle - \langle (Df)(x), b\rangle|du\\
&\leq \quad \frac{1}{t}\int_0^t ||(\mathcal{T}_u - I)(\langle J^*(Df)(\cdot), \cdot\rangle)||_K du\\
&+ \quad \frac{1}{t}\int_0^t ||(\mathcal{T}_u - I)(\langle (Df)(\cdot), S(u)b\rangle)||_K du\\
&+ \quad \frac{1}{t}\int_0^t ||\langle (Df)(\cdot), (S(u) - I)b\rangle||_K du\\
&\leq \quad \frac{1}{t}\int_0^t ||(\mathcal{T}_u - I)(\langle (Df)(\cdot), J(\cdot)\rangle)||_K du\\
&+ \quad \frac{1}{t}\int_0^t ||(\mathcal{T}_u - I)(\langle (Df)(\cdot), S(u)b\rangle)||_K dr\\
&+ \quad \frac{1}{t}\int_0^t ||(Df)(\cdot)||_K.||(S(u) - I)b||dr\\
&\quad\quad \to 0 \text{ as } t \to 0.
\end{aligned}
$$

Similar manipulations yield

$$
\begin{aligned}
2I_t^{(2)}(f) \quad &\leq \quad \frac{1}{t}\int_0^t ||(\mathcal{T}_u - I)\mathrm{tr}((D^2 f)(\cdot)S(u)QS(u)^*)||_K du\\
&+ \quad \frac{1}{t}\int_0^t ||\mathrm{tr}((D^2 f(\cdot)(S(u) - I)QS(u)^*)||_K du\\
&+ \quad \frac{1}{t}\int_0^t ||\mathrm{tr}((D^2 f(\cdot)S(u)Q(S(u)^* - I))||_K du\\
&\quad\quad \to 0 \text{ as } t \to 0.
\end{aligned}
$$

$$
\begin{aligned}
I_t^{(3)}(f) \quad &\leq \quad \frac{1}{t}\int_0^t \int_{||y||\geq 1} ||(\mathcal{T}_u - I)[f(\cdot + S(u)y) - f(\cdot)]||_K \nu(dy)du\\
&+ \quad \frac{1}{t}\int_0^t \int_{||y||\geq 1} ||(\mathcal{T}_u[f(\cdot + S(u)y) - f(\cdot + y)]||_K \nu(dy)du\\
&\quad\quad \to 0 \text{ as } t \to 0.
\end{aligned}
$$

Using Taylor's theorem we obtain,

$$
I_t^{(4)}(f) \quad \leq \quad \frac{1}{t}\int_0^t \int_{||y||<1} ||(\mathcal{T}_u - I)(\langle S(u)y, D^2 f(\cdot)S(u)y\rangle)||_K \nu(dy)du
$$

$$
\begin{aligned}
&+ \quad \frac{1}{t}\int_0^t \int_{||y||<1} ||\langle (S(u)-I)y, D^2 f(\cdot)S(u)y\rangle||_K \nu(dy)du \\
&+ \quad \frac{1}{t}\int_0^t \int_{||y||<1} ||\langle S(u)y, D^2 f(\cdot)(S(u)-I)y\rangle||_K \nu(dy)dr \\
&\quad \to 0 \text{ as } t \to 0.
\end{aligned}
$$

Collecting together these results, we deduce that

$$\limsup_{t\to 0}\left|\left|\frac{1}{t}(\mathcal{T}_t f - f) - \mathcal{L}f\right|\right|_K = 0,$$

Hence $\mathcal{A}_Y|_{C_J^2(H)} = \mathcal{L}$. □

Note that, in the case where the Lévy process X has an invariant measure μ, an L^2-version of (4.17) can be found in Proposition 3.5 of [24].

We now show that our formula for the generator is consistent with the pseudo-differential operator representation constructed in [23].

Let $\mathcal{F}S_J(H)$ be the subspace of $\mathcal{F}C_J^2(H)$ comprising those cylinder functions for which f is a Schwartz function. By the argument in section 1 of [23] we can assert that each $f \in \mathcal{F}S_J(H)$ is the Fourier transform of a complex signed measure ν on the Borel σ-algebra of H which has finite total variation, i.e.

$$f(x) = \int_H e^{i\langle x,\xi\rangle}\nu(d\xi),$$

for all $x \in H$.

Proposition 4.1 (c.f. [23]) *For all $f \in \mathcal{F}S_J(H), x \in Dom(J)$,*

$$(\mathcal{A}_Y f)(x) = \int_H (i\langle Jx, \xi\rangle - \eta(\xi))e^{i\langle x,\xi\rangle}\nu(d\xi).$$

Proof. This follows easily from the facts that

$$(A_X f)(x) = -\int_H \eta(\xi)e^{i\langle x,\xi\rangle}\nu(d\xi),$$

which is established as in the "classical" finite dimensional case (c.f. [2], section 3.3.2) and for each fixed $\xi \in H$,

$$De^{i\langle\cdot,\xi\rangle} = ie^{i\langle\cdot,\xi\rangle}\xi. \qquad \square$$

The result of Proposition 4.1 extends easily to the action of L on $L^p(H,\mu)$ where $p \geq 1$ in the case where the process Y has an invariant measure μ (c.f. [23]).

Since Mehler semigroups are quasi-equicontinuous, it follows from proposition 3.1(iii) that its generator is closed as an operator from $(C_b(H), \tau_{uc})$ to $(C_b(H), ||\cdot||)$.

5 The Generator in The Mixed Topology

We recall that sequential convergence takes place in the mixed topology τ_m if and only if we can verify M(i) and M(ii) as given at the beginning of section 4. In this section, we consider $(\mathcal{T}_t, t \geq 0)$ as a semigroup in $(C_b(H), \tau_m)$. It is strongly continuous by theorem 4.1, indeed M(i) is established therein and M(ii) is trivial (see also theorem 6.2 in [16]). We denote the infinitesimal generator of $(\mathcal{T}_t, t \geq 0)$ in $(C_b(H), \tau_m)$ by $\mathcal{A}_Y^{(m)}$. It is densely defined by general considerations (see e.g. proposition 1.3 in [19]). In fact it is shown in [14] that the semigroup is locally equicontinuous and hence $\mathcal{A}_Y^{(m)}$ is closed. We given an independent proof of this latter fact below for the convenience of the reader.

Theorem 5.1 *1. $\mathcal{A}_Y^{(m)}$ is closed.*

2. $C_J^2(H) \subseteq Dom(\mathcal{A}_Y^{(m)})$ and $\mathcal{A}_Y^{(m)} f = \mathcal{L}f$ for all $f \in C_J^2(H)$.

Proof.

1. We argue as in the proof of proposition 3.1.

   Let $(f_n, n \in \mathbb{N})$ be a sequence of vectors in $C_b(H)$ for which $\tau_m - \lim_{n\to\infty} f_n = f$ and $\tau_m - \lim_{n\to\infty} \mathcal{A}_Y^{(m)} f_n = g \in C_b(H)$. For each $t \geq 0$,

$$\mathcal{T}_t f - f = \tau_m - \lim_{n\to\infty} \int_0^t \mathcal{T}_s \mathcal{A}_Y^{(m)} f_n ds.$$

   We have

$$\begin{aligned} \sup_{n\in\mathbb{N}} \left|\left| \int_0^t \mathcal{T}_s \mathcal{A}_Y^{(m)} f_n ds \right|\right|_0 &\leq \int_0^t \sup_{n\in\mathbb{N}} ||\mathcal{T}_s \mathcal{A}_Y^{(m)} f_n||_0 ds \\ &\leq t \sup_{n\in\mathbb{N}} ||\mathcal{A}_Y^{(m)} f_n||_0 < \infty \end{aligned}$$

   We need to show that

$$\tau_m - \lim_{n\to\infty} \int_0^t \mathcal{T}_s \mathcal{A}_Y^{(m)} f_n ds = \int_0^t \mathcal{T}_s g ds.$$

To establish M(i), we see that for each compact $K \subseteq H$,

$$\begin{aligned} & \lim_{n\to\infty} \sup_{x\in K} \left| \int_0^t \mathcal{T}_s \mathcal{A}_Y^{(m)} f_n(x) ds - \int_0^t \mathcal{T}_s g(x) ds \right| \\ \leq & \lim_{n\to\infty} \int_0^t \sup_{x\in K} |\mathcal{T}_s(\mathcal{A}_Y^{(m)} f_n(x) - g(x))| ds \\ \to & \ 0 \text{ as } n \to \infty, \end{aligned}$$

by the τ_{uc}-continuity of the semigroup and dominated convergence, where we have used the estimate derived above. The same estimate allows us to establish M(ii) and the result follows.

2. By the result of theorem 4.2, we have

$$\lim_{t\to 0} \sup_{x\in K} \left| \frac{1}{t}[(\mathcal{T}_t f)(x) - f(x)] - (\mathcal{L}f)(x) \right| = 0,$$

for all $f \in C_J^2(H)$ and compact $K \subseteq H$, so we need only show that each

$$K(f) := \sup_{t>0} \left|\left| \frac{1}{t}(\mathcal{T}_t f - f) - \mathcal{L}f \right|\right|_0 < \infty.$$

However

$$\begin{aligned} K(f) &= \sup_{t>0} \sup_{x\in H} \left| \frac{1}{t} \int_0^t [(\mathcal{T}_t - I)\mathcal{L}f](x) ds \right| \\ &\leq 2 \sup_{x\in H} |(\mathcal{L}f)(x)| < \infty, \end{aligned}$$

and the result follows. □

Proposition 5.1 *Let $D \subseteq Dom(\mathcal{M})$ be a τ_m-dense linear manifold in $C_b(H)$ such that $\mathcal{T}_t(D) \subseteq D$ for all $t \geq 0$, then D is a core for $\mathcal{A}_Y$.*

Proof. This is established along classical lines as in lemma 4.4 in [15]. □

Before we can apply Proposition 5.1, we need a technical lemma. First some notation. For each $x \in H, t \geq 0$, define

$$Y_x(t) := S(t)x + \int_0^t S(t-s)CdX(s),$$

$$\text{and } \widetilde{Y_x}(t) := S(t)x + \int_0^t S(t-s)Cd\tilde{X}(s),$$

where $\tilde{X}(t) = X(t) - \int_{B^c} uN(t, du)$.

Lemma 5.1 *If $\int_{B^c}||Cu||\nu(du) < \infty$, then $\mathbb{E}(|\langle w, Y_x(t)\rangle|) < \infty$ for all $t \geq 0, x, w \in H$.*

Proof. First observe that

$$\mathbb{E}(|\langle w, \widetilde{Y_x}(t)\rangle|) \leq ||w||\mathbb{E}(||\widetilde{Y_x}(t)||) \leq ||w||\mathbb{E}(||\widetilde{Y_x}(t)||^2)^{\frac{1}{2}} < \infty,$$

using the Itô isometry as in the proof of Lemma 2.1 (i). It follows that $\mathbb{E}(|\langle w, Y_x(t)\rangle|)$ is finite iff $\mathbb{E}\left(\left|\langle w, \int_0^t \int_{B^c} S(t-s)CuN(du, ds)\rangle\right|\right)$ is finite. However

$$\begin{aligned}
\mathbb{E}\left(\left|\langle w, \int_0^t \int_{B^c} S(t-s)CuN(du, ds)\rangle\right|\right) &\leq \mathbb{E}\left(\int_0^t \int_{B^c} |\langle w, S(t-s)Cu\rangle| N(du, ds)\right) \\
&= \int_0^t \int_{B^c} |\langle w, S(t-s)Cu\rangle|\nu(du)ds \\
&\leq ||w|| \int_0^t ||S(t-s)||ds. \int_{B^c} ||Cu||\nu(du) \\
&\leq \frac{M}{\beta}|e^{\beta t} - 1| \int_{B^c} ||Cu||\nu(du) < \infty,
\end{aligned}$$

where we have used (2.10), taking $\beta \neq 0$ without loss in generality. □

The next theorem generalises a result obtained in [15], theorem 4.5 for the case of Ornstein-Uhlenbeck processes driven by Brownian motion.

Theorem 5.2 *If $\int_{B^c}||Cu||\nu(du) < \infty$, then $\mathcal{FC}_J^2(H)$ is a core for $\mathcal{A}_Y$.*

Proof. By proposition 5.1, we must show that $\mathcal{T}_t(\mathcal{FC}_J^2(H)) \subseteq \mathcal{FC}_J^2(H)$, for all $t \geq 0$. If F_f is as in (4.16), then for all $x \in H, t \geq 0$,

$$\begin{aligned}
(\mathcal{T}_t F_f)(x) &= \int_H f(\langle x_1, S(t)x + y\rangle, \ldots, \langle x_n, S(t)x + y\rangle)\rho_t(dy) \\
&= \int_{F_n} f(\langle S(t)^* x_1, x\rangle + \langle x_1, y\rangle, \ldots, \langle S(t)^* x_n, x\rangle + \langle x_n, y\rangle)\rho_t^{F_n}(dy),
\end{aligned}$$

where $\mu_t^{F_n}$ is the restriction of μ_t to the finite dimensional vector space $F_n :=$ lin.span$\{x_1, \ldots, x_n\}$. Hence $\mathcal{T}_t F_f = F_{g_{f,t}}$ where

$$g_{f,t}(z) := \int_{F_n} f(z_1 + \langle x_1, y\rangle, \ldots, z_n + \langle x_n, y\rangle)\rho_t^{F_n}(dy),$$

for each $z = (z_1, \ldots, z_n) \in \mathbb{R}^n$.

Furthermore, for each $x \in H$

$$\langle x, J^*(D\mathcal{T}_t F_f)(x)\rangle = \sum_{i=1}^n \int_H (\partial_i f)(\langle x_1, S(t)x+y\rangle, \ldots, \langle x_n, S(t)x+y\rangle)\rho_t(dy)\langle x, J^* S(t)^* x_i\rangle,$$

hence by lemma 5.1

$$\begin{aligned}
& \sup_{x\in H} |\langle x, J^*(D\mathcal{T}_t F_f)(x)\rangle| \\
\leq\; & \sup_{x\in H} \left| \sum_{i=1}^n \int_H (\partial_i f)(\langle x_1, S(t)x+y\rangle, \ldots, \langle x_n, S(t)x+y\rangle)\langle S(t)x+y, J^* x_i\rangle \rho_t(dy)\right| \\
+\; & \sup_{x\in H} \left| \sum_{i=1}^n \int_H (\partial_i f)(\langle x_1, S(t)x+y\rangle, \ldots, \langle x_n, S(t)x+y\rangle)\langle y, J^* x_i\rangle \rho_t(dy)\right| \\
\leq\; & \sup_{x\in H} |\langle x, J^*(DF_f)(x)\rangle| + n \max_{1\leq i\leq n} \sup_{z\in\mathbb{R}^n} (|(\partial_i f)(z)| \int_H |\langle y, J^* x_i\rangle| \rho_t(dy) \\
<\; & \infty. \qquad \square
\end{aligned}$$

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