Pseudo Differential Operators and Markov Processes in Lie Groups

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Markov Processes and Symbols

Let $(X(t), t \geq 0)$ be a Markov process defined a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), P)$ taking values in a locally compact space E so that for all $f \in B_b(E), s \leq t$

$$\mathbb{E}(f(X(t))|\mathcal{F}_s) = \mathbb{E}(f(X(t))|X(s)) \text{ a.s.}$$

Define linear operators $(T_t, t \geq 0)$ on $B_b(E)$ by

$$T_t f(x) = \mathbb{E}(f(X(t))|X(0) = x).$$

Positivity $f \geq 0 \Rightarrow T_t f \geq 0$

Semigroup property $T_{s+t} = T_s T_t$

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So it has a closed, densely defined infinitesimal generator A.

For all $f \in \text{Dom}(A)$

$$\lim_{t \to 0} \left|\left| \frac{T_t t - f}{t} - Af \right|\right| = 0.$$

Let $E = \mathbb{R}^d$

Key Questions (Niels Jacob)

1. When can we write A and T_t as pseudo-differential operators? So if $f \in C_c^\infty(\mathbb{R}^d) \subseteq \text{Dom}(A)$:

$$Af(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{ix \cdot \xi} q(x, \xi) \widehat{f}(\xi) d\xi$$

2. Can we extract interesting probabilistic information from the symbol q ?

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In particular, if we have a symbol q with reasonable properties, can we construct a suitably regular Markov process on $\mathbb{R}^d$ for which the generator is a pseudo-differential operator with symbol q?

Techniques for achieving this programme:

- Construction of a (symmetric) Dirichlet form to obtain associated Hunt process.
- Direct construction of the Feller semigroup using the Hille-Yosida-Ray theorem.
- Solve the martingale problem.

See works by Jacob, Schilling, Hoh, ...

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Jacob's Probabilistic Interpretation of the Symbol

If the process has a symbol q then

$$\begin{aligned} q(x,\xi) &= \lim_{t\to 0}\frac{\mathbb{E}(e^{i\xi\cdot(X(t)-x)}|X(0)=x)-1}{t} \\ &= e^{-i\xi\cdot x}\lim_{t\to 0}\frac{T(t)e^{i\xi\cdot x}-e^{i\xi\cdot x}}{t} \\ &= e^{-i\xi\cdot x}A(e^{i\xi\cdot})(x) \end{aligned}$$

if $e^{i\xi\cdot}\in\text{Dom}(A)$.

see N.Jacob, *Pseudo Differential Operators and Markov Processes: 3, Markov Processes and Applications*, World Scientific (2005).

Representations of Compact Lie Groups

Take E to be a compact Lie group G.

If H is a complex separable Hilbert space then $\mathcal{U}(H)$ is the group of all unitary operators on H.

A *unitary representation* of a locally compact group G is a strongly continuous homomorphism π from G to $\mathcal{U}(V_\pi)$. So we have for all $g, h \in G$:

$$\pi(gh) = \pi(g)\pi(h),$$

$$\pi(e) = I_\pi,$$

$$\pi(g^{-1}) = \pi(g)^{-1} = \pi(g)^*.$$

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π is *irreducible* if we cannot write $\pi(g) = (\pi_1(g), \pi_2(g))$ for all g where π_1, π_2 are representations on closed subspaces of V_π.

Trivial representation δ on $\mathbb{C} : \delta(g) = 1$ for all g.

The *unitary dual* of G, $\widehat{G}$ is set of (equivalence classes of) all irreducible representations of G.

As G compact, $\widehat{G}$ is countable

and for all $\pi \in \widehat{G}, d_\pi := \dim(V_\pi) < \infty \Rightarrow$ each $\pi(g)$ is a unitary matrix.

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Define $h_\pi(\psi, \phi)(g) := \langle \pi(g)\psi, \phi \rangle$.

Define $\mathcal{M} :=$ lin. span$\{h_\pi(\psi, \phi); \psi, \phi \in V_\pi, \pi \in \widehat{G}\}$.

Let $L^2(G, \mathbb{C}) := L^2(G, m, \mathbb{C})$ where m is normalised (bi-invariant) Haar measure on G.

Theorem 1

[Peter-Weyl] $\mathcal{M}$ is dense in $L^2(G, C)$ and also in $C(G)$ (equipped with the uniform topology.) Furthermore the set $\{d_\pi^{\frac{1}{2}}\pi_{ij}, 1 \leq i, j \leq d_\pi, \pi \in \widehat{G}\}$ is a complete orthonormal basis for $L^2(G, \mathbb{C})$.

It follows that we have the *Fourier expansion* of $f \in L^2(G, \mathbb{C})$:

$$f = \sum_{\pi \in \widehat{G}} d_\pi \mathrm{tr}(\widehat{f}(\pi)\pi),$$

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Notation: Let $M_n(\mathbb{C})$ be the space of $n \times n$ matrices with complex coefficients.

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Pseudo Differential Operators on Compact Groups

The aim is to extend the definition of pseudo-differential operators of *Ruzhansky/Turunen* so as to obtain a tool with the same probabilistic value as *Jacob* developed in $\mathbb{R}^d$.

The following definition was given in DBA *J. Math. Anal. Appl.* **384**, 331-48 (2011):

We say that a linear operator A defined on $L^2(G, \mathbb{C})$ is a *pseudo differential operator with symbol* σ_A if

(PD1) $\mathcal{M} \subseteq \text{Dom}(A)$,

(PD2) There exists a mapping $\sigma_A : G \times \widehat{G} \to \mathcal{R}(\widehat{G})$ such that $\sigma_A(g, \pi) \in M_{d_\pi}(\mathbb{C})$ for each $g \in G, \pi \in \widehat{G}$ with $\sigma_A(g, \pi) = \pi(g^{-1})A\pi(g)$ for all $\pi \in \widehat{G}, g \in G$,

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(PD3) For each $\pi \in \widehat{G}$, the mapping $g \to \sigma_A(g, \pi)$ is weakly (or equivalently, strongly) measurable.

The key condition $\sigma_A(g, \pi) = \pi(g^{-1})A\pi(g)$ in (PD2) is shorthand for

$$\sigma_A(g, \pi)_{ij} = \sum_{k=1}^{d_\pi} \overline{\pi_{ki}(g)} A\pi_{kj}(g), \tag{1.1}$$

for $1 \leq i, j \leq d_\pi$.

We can recover the "Fourier inversion representation" of the operator A in our theory as follows:

Theorem 3

Let A be a pseudo differential operator on $L^2(G, \mathbb{C})$ with symbol σ_A. Suppose that A is closed and that $\sum_{\pi \in \widehat{G}} d_\pi \text{tr}(\widehat{f}(\pi) A\pi)$ converges in $L^2(G, \mathbb{C})$ for all $f \in \text{Dom}(A)$. Then

$$Af(g) = \sum_{\pi \in \widehat{G}} d_\pi \text{tr}(\sigma_A(g, \pi) \widehat{f}(\pi) \pi(g)), \tag{1.2}$$

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Proof. By Fourier expansion we have for $f \in \text{Dom}(A)$

$$f = \sum_{\pi \in \widehat{G}} d_\pi \sum_{i,j=1}^{d_\pi} \widehat{f}(\pi)_{ij} \pi_{ji}.$$

It follows from the fact that A is closed that for almost all $g \in G$,

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The following result gives an equivalent characterisation of a pseudo differential operator:

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A linear operator A defined on $L^2(G, \mathbb{C})$ with $\mathcal{M} \subseteq \text{Dom}(A)$ is a pseudo differential operator if and only if there exists a mapping $\sigma_A : G \times \widehat{G} \rightarrow \mathcal{R}(\widehat{G})$ satisfying (PD3) such that $\sigma_A(g, \pi) \in M_{d_{pi}}(\mathbb{C})$ and

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The converse follows when we take ψ and ϕ to be orthonormal basis vectors $e_i^{(\pi)}$ and $e_j^{(\pi)}$ (respectively). We then find that

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and so

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By unitarity of the matrix $\pi(g)$ for each $1 \leq l \leq d_\pi$, we obtain

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If G is not compact, it has at least one infinite dimensional irreducible representation. If $\pi \in \widehat{G}$, replace $M_{d_\pi}(\mathbb{C})$ with $\mathcal{L}(V_\pi)$, the space of all bounded linear operators on the complex separable Hilbert space V_π.

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Pseudo Differential Operators on Lie Groups

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Ingredients for extension of pseudo-differential operators:

- *Differential Representations*

  Let $\mathfrak{g}$ be the Lie algebra of G and $\exp : \mathfrak{g} \rightarrow G$ be the exponential map.

  For each $\pi \in \widehat{G}$ obtain a Lie algebra representation $d\pi$ by

$$\pi(\exp(tX)) = e^{td\pi(X)},$$

  By Stone's theorem, each $d\pi(X)$ is a skew-adjoint linear operator on V_π.

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  (Harish-Chandra, Cartier-Dixmier, Nelson.)

  The space of *analytic vectors* for $\pi \in \widehat{G}$ is

$$V_\pi^\omega := \{\psi \in V_\pi, g \to \pi(g)\psi \text{ is analytic}\}.$$

Facts.

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Define $\mathcal{M} :=$ lin. span$\{h_\pi(\psi, \phi); \psi \in V_\pi^\omega, \phi \in V_\pi, \pi \in \widehat{G}\}$.

Note that $\mathcal{M} \subseteq C_b(G)$.

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We say that a linear operator A defined on $C_0(G)$ (and having a unique extension to $C_b(G)$) is a *pseudo differential operator with symbol* σ_A if

(PD1) $\mathcal{M} \subseteq \text{Dom}(A)$,

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The Symbol of a Convolution Semigroup

The *convolution* of probability measures $\mu_1 * \mu_2$ is the unique probability measure for which

$$\int_G f(\sigma)(\mu_1 * \mu_2)(d\sigma) = \int_G \int_G f(\sigma\tau)\mu_1(d\sigma)\mu_2(d\tau),$$

for all bounded Borel functions f.

Let $(\mu_t, t \geq 0)$ be a weakly continuous convolution semigroup on G. So

- $\mu_0 = \delta_e$.
- $\mu_{t+s} = \mu_t * \mu_s$ for all $s, t \geq 0$.

If $(X_t, t \geq 0)$ is a G-valued Lévy process then $\mu_{t-s} = \text{Law}(X_{t-s}) = \text{Law}(X_s^{-1} X_t)$.

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Obtain a C_0-semigroup $(T_t, t \geq 0)$ with generator $\mathcal{L}$ on $C_0(G)$ by

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$C^2(G) := \{f \in C_0(G); X_i(f) \in C_0(G)$ and $X_iX_j(f) \in C_0(G)$ for all $1 \leq i, j \leq d\}$ is dense in $C_0(G)$.

There exist functions $x_i \in C_c^\infty(G), 1 \leq i \leq d$ so that $x_i(e) = 0, X_i x_j(e) = \delta_{ij}$ and $(x_1, \ldots, x_d)$ are canonical co-ordinates in a neighbourhood U of e.

A measure ν defined on $\mathcal{B}(G - \{e\})$ is called a *Lévy measure* whenever

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Theorem 4

1. $C^2(G) \subseteq \text{Dom}(\mathcal{L})$.
2. For each $g \in G, f \in C^2(G)$,

$$\begin{aligned} \mathcal{L}f(g) &= b^i X_i f(g) + a^{ij} X_i X_j f(g) \\ &+ \int_{G-\{e\}} (f(g\tau) - f(g) - x^i(\tau) X_i f(g)) \nu(d\tau), \end{aligned} \quad (1.4)$$

where $b = (b^1, \ldots b^d) \in \mathbb{R}^d$, $a = (a^{ij})$ is a non-negative-definite, symmetric $d \times d$ real-valued matrix and ν is a Lévy measure on $G - \{e\}$.

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The following is essentially proved (without mention of pseudo differential operators) in DBA, *J. Theor. Prob.* **14**, 61-76 (2001).

Theorem 5

$\mathcal{L}$ is a pseudo differential operator with symbol

$$\begin{aligned} \sigma(g,\pi) &= b^i d\pi(X_i) + a^{ij} d\pi(X_i) d\pi(X_j) \\ &+ \int_{G-\{e\}} [\pi(\tau) - I_\pi - x^i(\tau) d\pi(X_i)] \nu(d\tau) \end{aligned} \tag{1.5}$$

To see how the proof works, its instructive to consider the *drift*, *diffusion* and *jump* parts of $\mathcal{L}$ separately.

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$$\begin{aligned} Xh_\pi(\psi,\phi)(g) &= \frac{d}{du}\langle \pi(g)\pi(\exp(uX)\psi,\phi\rangle\Big|_{u=0} \\ &= \frac{d}{du}\langle e^{ud\pi(X)}\psi,\pi(g^{-1})\phi\rangle\Big|_{u=0} \\ &= \langle \pi(g)d\pi(X)\psi,\phi\rangle. \end{aligned}$$

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Lévy-type Operators

Generalise the linear operators obtained in Hunt's theorem to consider

$$\mathcal{L}f(g) = b^i(g)X_i f(g) + a^{ij}(g)X_i X_j f(g) + \int_G [f(\tau) - f(g) - x^i(g^{-1}\tau)X_i f(g)]\nu(g, d\tau), \qquad (1.6)$$

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If the mapping $g \to \int_G x_i(g^{-1}\tau)x_j(g^{-1}\tau)h^{ij}(g)\nu(g, d\tau)$ is continuous for all $h^{ij} \in C_0(G)$,

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Under strong assumptions, I have associated a symmetric Dirichlet form to $\mathcal{L}$. In particular this works when $\mathcal{L} = \mathcal{L}_c + \mathcal{L}_d$ where

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We may also attempt to associate a Feller semigroup and hence a Feller process to $\mathcal{L}$ by verifying the conditions of the *Hille-Yosida-Ray theorem.*

As $\mathcal{L}$ is densely defined we need to establish that

1. $\mathcal{L}$ satisfies the *positive maximum principle*, i.e. $f \in \mathrm{D}om(\mathcal{L})$ and $f(\sigma_0) = \sup_{\sigma \in G} f(\sigma) \geq 0 \Rightarrow \mathcal{L}f(\sigma_0) \leq 0$.
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Elliptic Operators

(see *D.W.Robinson "Elliptic Operators and Lie Groups", OUP (1991)*.)

Recall the basis $(X_1, \ldots, X_d)$ of $\mathfrak{g}$.

Consider the differential operator of order m
$X := \sum_{|\alpha| \leq m} c_\alpha(\cdot) X^\alpha$,

where $X^\alpha := X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_d^{\alpha_d}$,

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Its *symbol* is an unbounded operator in V_π with domain V_π^ω:

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Some nice estimates for second order strongly elliptic second order operators include Gårding's inequality, heat kernel bounds. A good strategy may be to study Lévy type operators that are *comparable* with strongly elliptic ones?

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