Lévy-type Stochastic Integrals with Regularly Varying Tails*

David Applebaum
Probability and Statistics Department,
University of Sheffield,
Hicks Building, Hounsfield Road,
Sheffield, England, S3 7RH

e-mail: D.Applebaum@sheffield.ac.uk

Abstract

Lévy-type stochastic integrals $M = (M(t), t \geqslant 0)$ are obtained by integrating suitable predictable mappings against Brownian motion B and an independent Poisson random measure N. We establish conditions under which the right tails of M are of regular variation. In particular we require that the intensity measure associated to N is the product of a regularly varying Lévy measure with Lebesgue measure. Both univariate and multivariate versions of the problem are considered.

Keywords and phrases: *Lévy-type stochastic integral, predictable mapping, semimartingale, regular variation, Lévy measure.*

1 Introduction

The noise driving a random dynamical system is typically taken to be a semimartingale. For many models of interest in the physical sciences, finance, insurance and telecommunications, probability laws with "heavy tails" are encountered, and this signifies that there is a significant probability that a

*Work carried out at The Nottingham Trent University

large scale discontinuity will interrupt the usual regular behaviour. Mathematically this is modelled by assuming that the tails are of regular variation (i.e. asymptotically like a Pareto distribution), or more generally subexponential (see e.g. [8], [20] for many practical examples).
This suggests that there should be considerable interest in semimartingales with regularly varying or subexponential tails, but there appears to have been little work on this subject so far. One case that has been intensively studied is that of a Lévy process. Here it is known that the tail of the process is subexponential (of regular variation) if and only if the tail of the Lévy measure (which determines the "large jumps" of the process) is subexponential (of regular variation) - see [5], [7]. Another paper of related interest to this one is [19] where conditions are found for the solutions of certain stochastic differential equations, which are driven by a regularly varying Lévy process, to themselves have a regularly varying right tail.
In this paper we study a class of semimartingales whose structure is particularly transparent. These are the Lévy-type stochastic integrals which are built from the noise of a Lévy process and a quadruple of suitable predictable mappings which are coupled to the drift, diffusion, small jumps and large jumps, respectively. A comprehensive account of such integrals can be found in [2].
Our main aim in this paper is to find conditions under which these integrals have regularly varying right tails and that moreover, this is entirely due to the effect of the large jumps. We do not claim that the conditions we impose are optimal, and they are surely far from necessary - indeed one of the main aims of this article is to stimulate more research into these and related problems. An intriguing potential application is the modelling of stock returns where regularly varying tails are observed in empirical studies (see e.g. [1], [14] and section 7.6 in [8]).
In the first four sections of this paper, we study the problem within a univariate setting. The final section sketches a multivariate extension making extensive use of a recent new approach to multivariate regular variation developed by Filip Lindskog [12]. The results here are of similar type to those in one-dimension, but of necessity, somewhat cruder.

Notation. $\mathbb{R}^+ = [0, \infty)$. Throughout this article $\hat{B} = (-1, 0) \cup (0, 1)$ and $\hat{B}^c = (-\infty, -1] \cup [1, \infty)$. Given any Borel set E in $\mathbb{R}, \mathcal{B}(E)$ denotes the σ-algebra of all Borel subsets of E, where E is equipped with the usual relative topology. If $M = (M(t), t \geqslant 0)$ is a càdlàg real-valued semimartingale, then $[M, M](t)$ denotes its quadratic variation at time t, so that

$$[M, M](t) = M(t)^2 - M(0)^2 - 2\int_0^t M(s-)dM(s).$$

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2 Lévy-type Stochastic Integrals

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geqslant 0), P)$ be a stochastic base wherein the filtration $(\mathcal{F}_t, t \geqslant 0)$ satisfies the usual hypotheses of completeness and right continuity. Let $B = (B(t), t \geqslant 0)$ be a standard one-dimensional Brownian motion and $N = (N(t, \cdot), t \geqslant 0)$ be a Poisson random measure on $\mathbb{R}^+ \times (\mathbb{R} - \{0\})$ which is independent of B. We assume that the intensity measure associated to N is of the form Leb$\otimes\nu$ where Leb denotes Lebesgue measure on $\mathbb{R}^+$ and ν is a Lévy measure on $\mathbb{R} - \{0\}$, i.e. $\int_{\mathbb{R}-\{0\}}(|x|^2 \wedge 1)\nu(dx) < \infty$. We will further assume that ν has support on the whole of $\mathbb{R} - \{0\}$. We denote by $\tilde{N}$ the associated compensated measure, so that

$$\tilde{N}(t, A) = N(t, A) - t\nu(A),$$

for all $t \geqslant 0, A \in \mathcal{B}(\mathbb{R} - \{0\})$.
For given $A \in \mathcal{B}(\mathbb{R})$, let $\mathcal{P}_A$ denote the smallest σ-algebra with respect to which all mappings $\phi : \mathbb{R}^+ \times A \times \Omega \to \mathbb{R}$ satisfying (1) and (2) below are measurable.

1. For each $0 \leqslant t < \infty$, the mapping $(x, \omega) \to \phi(t, x, \omega)$ is $\mathcal{B}(E) \otimes \mathcal{F}_t$ measurable,

2. For each $x \in E, \omega \in \Omega$, the mapping $t \to \phi(t, x, \omega)$ is left continuous.

$\mathcal{P}_A$-measurable mappings are said to be A-predictable.
Now let (F, G, H, K) be a quadruple wherein F and G are predictable processes (in the usual sense) while H and K are $\hat{B}$- and $\hat{B}^c$-predictable mappings, respectively. We impose the assumption that for all $t \geqslant 0$,

$$\int_0^t \left(|F(s)| + |G(s)|^2 + \int_{\hat{B}} |H(s, x)|^2 \nu(dx) \right) ds < \infty \text{ a.s..}$$

We may then define the *Lévy-type stochastic integral* $M = (M(t), t \geqslant 0)$, where for each $t \geqslant 0$,

$$\begin{aligned} M(t) &:= \int_0^t F(s)ds + \int_0^t G(s)dB(s) + \int_0^t \int_{\hat{B}} H(s,x)\tilde{N}(ds,dx) \\ &+ \int_0^t \int_{\hat{B}^c} K(s,x)N(ds,dx) \\ &:= I_1^F(t) + I_2^G(t) + I_3^H(t) + I_4^K(t). \end{aligned} \tag{2.1}$$

M is a semimartingale. Both I_2^G and I_3^H are local martingales, while $I_1^F + I_4^K$ is a process of finite variation and each $I_4^K(t)$ is a (random) finite sum. M has a càdlàg modification, which is itself a semimartingale and which we identify with M henceforth. Further details of the construction and properties of M may be found in [2] (see also Chapter 4 of [10]).

Let $\mathcal{R}_\alpha$ denote the set of all regularly varying functions from $\mathbb{R}^+$ to $\mathbb{R}^+$ with index $\alpha \in \mathbb{R}$, so $f \in \mathcal{R}_\alpha$ if and only if

$$\lim_{x\to\infty} \frac{f(cx)}{f(x)} = c^\alpha, \quad \text{for all } c > 0.$$

Members of $\mathcal{R}_0$ are said to be slowly varying and $f \in \mathcal{R}_\alpha$ if and only if there exists $L \in \mathcal{R}_0$ such that $f(x) = L(x)x^\alpha$, for all $x \in \mathbb{R}^+$. Standard references for regular variation are [4], [16] and [21]. We will also want to discuss this type of limiting behaviour as $x \to -\infty$. Let $g : (-\infty, 0) \to \mathbb{R}^+$ and define $\tilde{g} : (0,\infty) \to \mathbb{R}^+$ by $\tilde{g}(x) = g(-x)$, for all $x > 0$. For each $\alpha \in \mathbb{R}$, we introduce the class $\mathcal{L}_\alpha$ by the prescription $g \in \mathcal{L}_\alpha$ if and only if $\tilde{g} \in \mathcal{R}_\alpha$.

Let F_X be the right continuous distribution function of a real-valued random variable X, and define $\overline{F_X} = 1 - F_X$. Let M be a Lévy-type stochastic integral. Our goal in this paper is to establish conditions under which $\overline{F_{M(t)}} \in \mathcal{R}_{-\alpha}$, for some $\alpha \geqslant 0$, and all $t \in [0,T]$. We will make the following standing assumption on the Lévy measure:-

$$\nu((-\infty,\lambda)) \in \mathcal{L}_{-\gamma} \text{ and } \nu((\lambda,\infty) \in \mathcal{R}_{-\alpha},$$

where $\alpha, \gamma \geqslant 0$.

We also require the following weak independence condition:- for each $t \geqslant 0, a \in \mathbb{R}$:

$$P(M(t) - I_4^K(t) > a | I_4^K(t) > b) \sim P(M(t) - I_4^K(t) > a), \tag{2.2}$$

as $b \to \infty$.

Before we begin the analysis, we mention one special case of interest. If $M(t) = \int_0^t F(s)dX(s)$, where X is an α-stable Lévy process , the stochastic integral $M = (M(t), t \geqslant 0)$ can be constructed so as to ensure that M automatically inherits regularly varying tails from X. Full details can be found in [18].

3 Some Elementary Tail Estimates

From now on we fix $T > 0$ and assume that $t \in (0, T]$. $C_1, C_2, \ldots$ will denote strictly positive constants. Our aim in this section is to establish conditions under which, for given $\alpha \geqslant 0$ and any $L \in \mathcal{R}_0$,

$$\lim_{\lambda\to\infty} \frac{P(|I_1^F(t)| \geqslant \lambda)}{\lambda^{-\alpha}L(\lambda)} = \lim_{\lambda\to\infty} \frac{P(|I_2^G(t)| \geqslant \lambda)}{\lambda^{-\alpha}L(\lambda)} = \lim_{\lambda\to\infty} \frac{P(|I_3^H(t)| \geqslant \lambda)}{\lambda^{-\alpha}L(\lambda)} = 0. \tag{3.3}$$

3.1 Estimates for I_1^F.

If $0 \leqslant \alpha < 1$, we assume that $\int_0^T \mathbb{E}(|F(s)|)ds < \infty$, then by Markov's inequality,

$$P(|I_1^F(t)| \geqslant \lambda) \leqslant \frac{\int_0^t \mathbb{E}|F(s)|)ds}{\lambda},$$

hence

$$\limsup_{\lambda\to\infty} \frac{P(|I_1^F(t)| \geqslant \lambda)}{\lambda^{-\alpha}L(\lambda)} = 0. \tag{3.4}$$

If $\alpha \geqslant 1$, we assume that $\int_0^T \mathbb{E}(|F(s)|^{\alpha+\epsilon})ds < \infty$ for some $\epsilon > 0$. By an easy Chebychev-type inequality and using Hölder's inequality, we obtain,

$$\begin{aligned} P(|I_1^F(t)| \geqslant \lambda) &\leqslant \frac{\mathbb{E}(|I_1^F(t)|^{\alpha+\epsilon})}{\lambda^{\alpha+\epsilon}} \\ &\leqslant \frac{t^{\alpha+\epsilon-1}\int_0^t \mathbb{E}(|F(s)|^{\alpha+\epsilon})ds}{\lambda^{\alpha+\epsilon}}, \end{aligned}$$

from which (3.4) follows immediately.

3.2 Estimates for I_2^G.

If $0 \leqslant \alpha < 2$, we assume that $\int_0^T \mathbb{E}(|G(s)|^2)ds < \infty$. Itô's isometry yields

$$\mathbb{E}(|I_2^G(t)|^2) = \int_0^t \mathbb{E}(|G(s)|^2)ds.$$

Arguing as above, we obtain

$$P(|I_2^G(t)| \geqslant \lambda) \leqslant \frac{\int_0^t \mathbb{E}(|G(s)|^2)ds}{\lambda^2},$$

from which the desired result follows.

If $\alpha \geqslant 2$, assume that $\int_0^T \mathbb{E}(|G(s)|^{\alpha+\epsilon})ds < \infty$, for some $\epsilon > 0$. As above, we have

$$P(|I_2^G(t)| \geqslant \lambda) \leqslant \frac{\mathbb{E}(|I_2^G(t)|^{\alpha+\epsilon})}{\lambda^{\alpha+\epsilon}}.$$

Using Burkholder and Hölder's inequalities, we obtain

$$\begin{aligned}
\mathbb{E}(|I_2^G(t)|^{\alpha+\epsilon}) &\leqslant C_1(\alpha,\epsilon)\mathbb{E}([I_2^G, I_2^G](t)^{\frac{\alpha+\epsilon}{2}}) \\
&= C_1(\alpha,\epsilon)\mathbb{E}\left[\left(\int_0^t |G(s)|^2 ds\right)^{\frac{\alpha+\epsilon}{2}}\right] \\
&\leqslant C_1(\alpha,\epsilon)t^{\frac{\alpha+\epsilon-2}{2}}\int_0^t \mathbb{E}(|G(s)|^{\alpha+\epsilon})ds,
\end{aligned}$$

and we are finished.

3.3 Estimates for I_3^H.

If $0 \leqslant \alpha < 2$, we assume that $\int_0^T \int_{\hat{B}} \mathbb{E}(|H(s,x)|^2)\nu(dx)ds < \infty$. In this case, Itô's isometry yields

$$\mathbb{E}(|I_3^H(t)|^2) = \int_0^t \int_{\hat{B}} \mathbb{E}(|H(s,x)|^2)\nu(dx)ds,$$

and we may proceed just as in the first part of section 3.2 above.
If $\alpha \geqslant 2$, we assume that $\int_0^T \int_{\hat{B}} \mathbb{E}(|H(s,x)|^{\alpha+\epsilon})\nu(dx)ds < \infty$, for some $\epsilon > 0$. We first suppose that $\nu(\hat{B}) < \infty$. Burkholder's inequality yields

$$P(|I_3^H(t)| > \lambda) \leqslant C_2(\alpha,\epsilon)\frac{\mathbb{E}([I_3^H, I_3^H](t)^{\frac{\alpha+\epsilon}{2}})}{\lambda^{\alpha+\epsilon}}. \tag{3.5}$$

$$\text{Let} \quad Y(t) := [I_3^H, I_3^H](t) = \int_0^t \int_{\hat{B}} H(s,x)^2 N(ds,dx).$$

Using Itô's formula, we obtain for $p > 1$:-

$$|Y(t)|^p = \int_0^t \int_{\hat{B}} (|Y(s-) + H(s,x)^2|^p - |Y(s-)|^p)N(ds,dx).$$

Taking expectations and using the elementary inequality, $|a+b|^p \leqslant 2^{p-1}(|a|^p + |b|^p)$, for $a, b \in \mathbb{R}$, we obtain:

$$\begin{aligned}
\mathbb{E}(|Y(t)|^p) &= \int_0^t \int_{\hat{B}} \mathbb{E}(|Y(s) + H(s,x)^2|^p - |Y(s)|^p)\nu(dx)ds \\
&\leqslant (2^{p-1} - 1)\nu(\hat{B})\int_0^t \mathbb{E}(|Y(s)|^p)ds + 2^{p-1}\int_0^T \int_{\hat{B}} \mathbb{E}(|H(s,x)|^{2p})\nu(dx)ds.
\end{aligned}$$

We now apply Gronwall's inequality to obtain

$$\mathbb{E}(|Y(t)|^p) \leqslant C_3(p,T) \int_{\hat{B}} \mathbb{E}(|H(s,x)|^{2p})\nu(dx)ds, \tag{3.6}$$

where $C_3(p,T) = 2^{p-1}\exp\{(2^{p-1}-1)\nu(\hat{B})T\}$. Now put $p = \frac{1}{2}(\alpha+\epsilon)$ in (3.6) and substitute into (3.5) to obtain the required result.
In the case where $\nu(\hat{B}) = \infty$, we must also assume that

$$\mathbb{E}\left[\left(\int_0^T \int_{\hat{B}} |H(s,x)|^2 \nu(dx)ds\right)^{\frac{\alpha+\epsilon}{2}}\right] < \infty. \tag{3.7}$$

We can then argue as above, using the following estimate, which is due to Kunita [11]:-

$$\begin{aligned} \mathbb{E}(|I_3^H(t)|^p) &\leqslant C_4(p)\left\{\int_0^t \int_{\hat{B}} \mathbb{E}(|H(s,x)|^p)\nu(dx)ds \right. \\ &+ \left. \mathbb{E}\left[\left(\int_0^t \int_{\hat{B}} |H(s,x)|^2\nu(dx)ds\right)^{\frac{p}{2}}\right]\right\}, \end{aligned}$$

for all $0 < t \leqslant T$.
Note that this estimate also holds in the case where $\nu(\hat{B}) < \infty$. A sufficient condition for (3.7) to hold is that there exists $h \in L^{\alpha+\epsilon}([0,T])$ such that $|H(s,x)| \leqslant |h(s)||x|$, for all $x \in \hat{B}, 0 \leqslant s \leqslant T$. This is easily verified by using Hölder's inequality and the definition of ν.
We will summarise all the results of this section within the statement of our main theorem in section 4.

4 The Main Theorem

We begin this section by investigating the asymptotic behaviour of the tail of $I_4^K(t) = \int_0^t \int_{\hat{B}^c} K(s,x)N(ds,dx)$. We make the following assumptions on the mapping K, for all $0 \leqslant t \leqslant T$.

1. For all $|x| \geqslant 1, K(t,x) = K(t)f(x)$ where K is a predictable process with $\inf_{0 \leqslant s \leqslant t} K(s) > 0$, for all $t > 0$ and $f : \hat{B}^c \to \mathbb{R}^+$ is Borel measurable.

2. $f_+ := f1_{\{x \geqslant 1\}} \in \mathcal{R}_\beta$ for some $\beta > 0$ and is non-decreasing with $\lim_{x\to\infty} f_+(x) = \infty$.

   $f_- := f1_{\{x \leqslant -1\}} \in \mathcal{L}_\delta$ for some $\delta > 0$ and is non-increasing with $\lim_{x\to-\infty} f_-(x) = \infty$.

Now define a process $Z_f = (Z_f(t), t \geqslant 0)$ by

$$Z_f(t) := \int_0^t \int_{\hat{B}^c} f(x)N(ds, dx) = \int_{\hat{B}^c} f(x)N(t, dx),$$

for each $t \geqslant 0$.

Proposition 4.1 *Assume that condition (2) above holds, then for each* $0 < t \leqslant T, \overline{F_{Z_f(t)}} \in \mathcal{R}_{-\rho}$, *where* $\rho = \min\left\{\frac{\alpha}{\beta}, \frac{\gamma}{\delta}\right\}$.

Proof. Z_f is a compound Poisson process with intensity measure ν_f where for each $A \in \mathcal{B}(\hat{B}^c), \nu_f(A) = \nu(f^{-1}(A))$ (see e.g. [2], Chapter 2). We write $Z_f(t) = Z_{f,1}(t) + Z_{f,2}(t)$, where $Z_{f,1}(t) = \int_{x \geqslant 1} f(x)N(t, dx)$ and $Z_{f,2}(t) = \int_{x \leqslant -1} f(x)N(t, dx)$ are independent compound Poisson processes.
Let $f_+^{\leftarrow}$ be the right continuous inverse of f_+, so that

$$f_+^{\leftarrow}(y) := \inf\{x \geqslant 1; f(x) > y\},$$

for each $y \in \mathbb{R}^+$, then for $\lambda > 0, \nu_{f_+}((\lambda, \infty)) = \nu \circ f_+^{\leftarrow}((\lambda, \infty))$. It follows from [16], Proposition 0.8 (v) and (iv) that $\overline{\nu_{f_+}} \in \mathcal{R}_{-\frac{\alpha}{\beta}}$. Hence by Proposition 0 in [7] (see also [9], pp. 572-3), it follows that $\overline{F_{Z_{f,1}(t)}} \in \mathcal{R}_{-\frac{\alpha}{\beta}}$ for all $0 < t \leqslant T$. A similar argument (where we define $f_-^{\leftarrow} := -(\tilde{f})^{\leftarrow}$) shows that $\overline{\nu_{f_-}} \in \mathcal{R}_{-\frac{\gamma}{\delta}}$, and hence $\overline{F_{Z_{f,2}(t)}} \in \mathcal{R}_{-\frac{\gamma}{\delta}}$ for all $0 < t \leqslant T$. We can now apply Theorem 6.1 in the appendix to deduce that each $\overline{F_{Z_f(t)}} \in \mathcal{R}_{-\rho}$.

□

Note: If we change assumption 2 above so that $f_- = 0$, then the conclusion of Proposition 4.1 will be that $\overline{F_{Z_f(t)}} \in \mathcal{R}_{-\frac{\alpha}{\beta}}$, for all $0 \leqslant t \leqslant T$. Similar remarks apply to the case where $f_+ = 0$.

We make some further assumptions:-

3. The processes $(K(t), t \geqslant 0)$ and $(Z_f(t), t \geqslant 0)$ are independent.

4. For each $0 \leqslant t \leqslant T$, there exists $\epsilon(t) > 0$ such that $\mathbb{E}(K(t)^{\rho+\epsilon(t)}) < \infty$.

If assumptions (1) to (4) hold, then it follows from Proposition 4.1 and section 4.2 of [15] (see also [20]) that $\overline{F_{J_4^{K,f}(t)}} \in \mathcal{R}_{-\rho}$, where $J_4^{K,f}(t) := K(t)Z_f(t)$, for each $t \geqslant 0$. We aim to develop this idea further. From now on assume that K is càglàd and define

$$\overline{K}(t) = \sup_{0 \leqslant s \leqslant t} K(s), \quad \underline{K}(t) = \inf_{0 \leqslant s \leqslant t} K(s),$$

for each $t \geqslant 0$.
We replace assumption (4) by the following stronger requirement:

4′. For each $0 \leqslant t \leqslant T$, there exists $\epsilon(t) > 0$ such that $\mathbb{E}(\overline{K}(t)^{\rho+\epsilon(t)}) < \infty$.

Theorem 4.1 *Assume that assumptions (1) to (3) and (4)′ hold, then* $\overline{F_{I_4^K(t)}} \in \mathcal{R}_{-\rho}$ *for each* $0 < t \leqslant T$.

Proof. Clearly each $\mathbb{E}(\underline{K}(t)^{\rho+\epsilon(t)}) < \infty$. Furthermore the processes $(\underline{K}(t), t \geqslant 0)$ and $(\overline{K}(t), t \geqslant 0)$ are each independent of $(Z_f(t), t \geqslant 0)$. Hence by the remarks made above, each of $\underline{K}(t)Z_f(t)$ and $\overline{K}(t)Z_f(t)$ have regularly varying right tail of index $-\rho$. Hence, given any $L \in \mathcal{R}_0$, for each $0 < t \leqslant T$ we have

$$\begin{aligned} 1 &= \lim_{\lambda\to\infty} \frac{P(\underline{K}(t)Z_f(t) > \lambda)}{\lambda^{-\rho}L(\rho)} \\ &\leqslant \liminf_{\lambda\to\infty} \frac{P(I_4^K(t) > \lambda)}{\lambda^{-\rho}L(\rho)} \leqslant \limsup_{\lambda\to\infty} \frac{P(I_4^K(t) > \lambda)}{\lambda^{-\rho}L(\rho)} \\ &\leqslant \lim_{\lambda\to\infty} \frac{P(\overline{K}(t)Z_f(t) > \lambda)}{\lambda^{-\rho}L(\rho)} = 1, \end{aligned}$$

and the required result follows. □

We can now construct some non-trivial examples of processes of the form I_4^k which have the required asymptotic behaviour. For example, we can take each $K(t) = g(B(t))$, where $B = (B(t), t \geqslant 0)$ is a standard Brownian motion and $g : \mathbb{R} \to (0, \infty)$ is continuous, convex and polynomially bounded. To see this, suppose that p is a polynomial such that $g(x) \leqslant p(|x|)$, for all $x \in \mathbb{R}$. Since g is convex, K is a submartingale, and so by Doob's martingale inequality, for all $0 \leqslant t \leqslant T$,

$$\begin{aligned} \mathbb{E}(\overline{K}(t)^{\rho+\epsilon(t)}) &\leqslant D(t,\rho)\mathbb{E}(K(t)^{\rho+\epsilon(t)}) \\ &\leqslant D(t,\rho)\mathbb{E}(p(|B(t)|^{\rho+\epsilon(t)}) < \infty, \end{aligned}$$

where $D(t,\rho) = (\frac{\rho+\epsilon(t)}{\rho+\epsilon(t)-1})^{\rho+\epsilon(t)}$ and we have chosen $\inf_{0\leqslant t\leqslant T}\epsilon(t) > (1-\rho)\vee 0$. Note that in order to satisfy our standing hypothesis of weak independence (2.2) (which we require in the next theorem), B should be chosen to be an independent copy of the process appearing in I_2^G.

We remark that Theorem 4.1 can be shown to hold under alternative constraints on K, when (4)′ is expressed in terms of the conditions for products of distribution functions to have regular variation which are described in the corollary to Theorem 3 of [6].

If we combine Theorem 4.1 with the estimates of section 3, we obtain our main result:-

Theorem 4.2 *Let* $M = (M(t), 0 \leqslant t \leqslant T)$ *be a Lévy-type stochastic integral of the form (2.1) satisfying the condition (2.2) . Suppose that the right tail of the Lévy measure* ν *is in* $\mathcal{R}_{-\alpha}$*, for some* $\alpha > 0$*, the left tail of* ν *is in* $\mathcal{L}_{-\gamma}$, *for some $\gamma > 0$ and that assumptions (1) to (3) and (4)′ above hold, for some $\beta, \delta > 0$. Let $\rho = \min\left\{\frac{\alpha}{\beta}, \frac{\gamma}{\delta}\right\}$. Further assume the following:-*

- *If $0 \leqslant \rho \leqslant 1$,*

$$\int_0^T \left[\mathbb{E}\left(|F(s)| + |G(s)|^2 + \int_{\hat{B}} |H(s,x)|^2 \nu(dx)\right)\right] ds < \infty.$$

- *If $1 \leqslant \rho < 2$, for some $\epsilon > 0$,*

$$\int_0^T \left[\mathbb{E}\left(|F(s)|^{\rho+\epsilon} + |G(s)|^2 + \int_{\hat{B}} |H(s,x)|^2 \nu(dx)\right)\right] ds < \infty.$$

- *If $\rho \geqslant 2$, for some $\delta_1, \delta_2, \delta_3 > 0$,*

$$\int_0^T \left[\mathbb{E}\left(|F(s)|^{\rho+\delta_1} + |G(s)|^{\rho+\delta_2} + \int_{\hat{B}} |H(s,x)|^{\rho+\delta_3} \nu(dx)\right)\right] ds < \infty,$$

  if $\nu(\hat{B}) < \infty$ or,

$$\begin{aligned} &\int_0^T \left[\mathbb{E}\left(|F(s)|^{\rho+\delta_1} + |G(s)|^{\rho+\delta_2} + \int_{\hat{B}} |H(s,x)|^{\rho+\delta_3} \nu(dx)\right)\right] ds \\ + \;\; &\mathbb{E}\left[\left(\int_0^T \int_{\hat{B}} |H(s,x)|^2 \nu(dx) ds\right)^{\frac{\rho+\delta_3}{2}}\right] < \infty, \end{aligned}$$

  if $\nu(\hat{B}) = \infty$.
  Then $\overline{F_{M(t)}} \in \mathcal{R}_{-\rho}$ for each $0 < t \leqslant T$.

Proof. Fix $0 < t \leqslant T$ and let $R(t) = M(t) - I_4^K(t)$. Then for each $0 < \eta < 1$,

$$\begin{aligned} P(M(t) > \lambda) &\leqslant P(I_4^K(t) > (1-\eta)\lambda) + P(R(t) > (1-\eta)\lambda) + P(I_4^K(t) \geqslant \eta\lambda, R(t) \geqslant \eta\lambda) \\ &\leqslant P(I_4^K(t) > (1-\eta)\lambda) + P(|R(t)| \geqslant (1-\eta)\lambda) + P(|R(t)| \geqslant \eta\lambda). \end{aligned}$$

Now for any $\kappa > 0$,

$$P(|R(t)| \geqslant \kappa) \leqslant P\left(|I_1^F(t)| \geqslant \frac{\kappa}{3}\right) + P\left(|I_2^G(t)| \geqslant \frac{\kappa}{3}\right) + P\left(|I_3^H(t)| \geqslant \frac{\kappa}{3}\right).$$

The results of section 3 ensure that (3.3) hold, hence

$$\lim_{\lambda\to\infty} \frac{P(|R(t)| \geqslant (1-\eta)\lambda) + P(|R(t)| \geqslant \eta\lambda)}{\lambda^{-\rho} L(\lambda)} = 0,$$

for any $L \in \mathcal{R}_0$. By Proposition 4.1 and the definition of regular variation, we deduce that

$$\limsup_{\lambda\to\infty} \frac{P(M(t)>\lambda)}{\lambda^{-\rho}L(\lambda)} \leqslant \lim_{\lambda\to\infty} \frac{P(I_4^K(t)>(1-\eta)\lambda)}{\lambda^{-\rho}L(\lambda)} = (1-\eta)^{-\rho}.$$

Now take limits as $\eta \downarrow 0$, to obtain

$$\limsup_{\lambda\to\infty} \frac{P(M(t)>\lambda)}{\lambda^{-\rho}L(\lambda)} \leqslant 1.$$

For the reverse inequality, fix $C > 0$, then

$$\begin{aligned} P(M(t)>\lambda) &\geqslant P(R(t)>-C, I_4^K(t)>\lambda+C) \\ &= P(R(t)>-C|I_4^K(t)>\lambda+C)P(I_4^K(t)>\lambda+C). \end{aligned}$$

By the assumption (2.2), we see that $P(R(t) > -C|I_4^K(t) > \lambda + C) \sim P(R(t) > -C)$, as $\lambda \to \infty$. Moreover, by the representation theorem for slowly varying functions ([4], section 1.3), it follows that $P(I_4^K(t) > \lambda+C) \sim P(I_4^K(t) > \lambda)$, as $\lambda \to \infty$. Hence we deduce that

$$\liminf_{\lambda\to\infty} \frac{P(M(t)>\lambda)}{\lambda^{-\rho}L(\lambda)} \geqslant P(R(t)>-C).$$

Now take limits as $C \to \infty$, to obtain $\liminf_{\lambda\to\infty} \dfrac{P(M(t)>\lambda)}{\lambda^{-\rho}L(\lambda)} \geqslant 1$, and the result follows. $\square$

5 The Multivariate Case

5.1 Multivariate Regular Variation

In this subsection, we give a brief overview of multivariate regular variation. Most of this is based on [12]. For further background on this topic, see e.g. [13], [17] and references therein.

Define $\overline{\mathbb{R}} = [-\infty, \infty]$. We note that $\overline{\mathbb{R}}^d - \{0\}$ can and will be equipped with a topology with respect to which it is a locally compact, complete, separable metric space. Furthermore under this topology, sets which are bounded away from the origin are relatively compact. The Borel σ-algebra of $\mathbb{R}^d - \{0\}$ under this topology coincides with that generated by the usual one. A Borel measure μ on $\overline{\mathbb{R}}^d - \{0\}$ is said to be *homogeneous of degree* $\alpha > 0$ if

$$\mu(cB) = c^{-\alpha}\mu(B),$$

for all $c > 0$ and all $B \in \mathcal{B}(\overline{\mathbb{R}}^d - \{0\})$.
Let ρ be a Borel measure defined on $\overline{\mathbb{R}}^d - \{0\}$. We say that it is *regularly varying* if there exists a sequence $(a_n, n \in \mathbb{N})$ with each $a_n > 0$ and $\lim_{n\to\infty} a_n = \infty$ and a Borel measure μ on $\overline{\mathbb{R}}^d - \{0\}$ for which $\mu(\overline{\mathbb{R}}^d - \mathbb{R}^d) = 0$ such that

$$n\rho(a_n\cdot) \xrightarrow{v} \mu(\cdot) \quad \text{as } n \to \infty \text{ on } \mathcal{B}(\overline{\mathbb{R}}^d - \{0\}), \tag{5.8}$$

where $\xrightarrow{v}$ indicates convergence in the vague topology.
It follows from this that the measure μ is homogeneous of degree α, for some $\alpha > 0$ and that the sequence $(a_n, n \in \mathbb{N})$ is regularly varying with index $\frac{1}{\alpha}$, in the sense that $\lim_{n\to\infty} \dfrac{a_{[cn]}}{a_n} = c^{\frac{1}{\alpha}}$, for all $c > 0$.
We use the terminology $\rho \sim RV(a_n, \mu)$ in this case. If X is a random variable taking values in $\overline{\mathbb{R}}^d - \{0\}$, we say that it is regularly varying if its law is regularly varying in the sense of (5.8). In this case we write $X \sim RV(a_n, \mu)$ and call μ the *limiting measure* of X. The link between the definition (5.8) and another well-known characterisation of multivariate regular variation is given in the following theorem (where $\xrightarrow{w}$ indicates convergence in the weak topology):-

Theorem 5.1 (Lindskog [12] theorem 1.15) *If X is a random variable taking values in $\mathbb{R}^d$ then $X \sim RV(a_n, \mu)$ if and only if there exists $\alpha > 0$ and a probability measure σ on $\mathcal{B}(\mathbb{S}^{d-1})$ such that*

$$\frac{P\left(|X| > cx, \dfrac{X}{|X|} \in \cdot\right)}{P(|X| > c)} \xrightarrow{w} x^{-\alpha}\sigma(\cdot),$$

as $c \to \infty$, for all $x > 0$.

(See also section 1 of Resnick [15] and references therein).

We now list some results which we will need to prove our main theorem:-

Theorem 5.2 *[Lindskog [12] theorem 1.28] If X_1 and X_2 are independent regularly varying random variables with limit measures μ_1 and μ_2, respectively, then $X_1 + X_2$ is regularly varying with limit measure $\mu_1 + \mu_2$.*

Theorem 5.3 *[Lindskog [12] theorem 1.30] If $(X_n, n \in \mathbb{N})$ is a sequence of i.i.d random variables with each $X_j \sim RV(a_n, \mu)$ and N is an independent Poisson random variable with intensity $c > 0$, then*

$$\sum_{j=1}^{N} X_j \sim RV(a_n, c\mu).$$

Theorem 5.4 *[Lindskog [12] lemma 1.32] If X is a random variable for which $\mathbb{E}(|X|^n) < \infty$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} nP(a_n^{-1}X \in B) = 0$, for every regularly varying sequence $(a_n, n \in \mathbb{N})$ and every relatively compact $B \in \mathcal{B}(\overline{\mathbb{R}}^d - \{0\})$.*

We say that a Borel measurable mapping $f : \mathbb{R}^d \to \mathbb{R}^m$ *respects origin exclusion* if whenever $B \in \mathcal{B}(\mathbb{R}^m - \{0\})$ is bounded below then so is $f^{-1}(B)$.

Theorem 5.5 *[Lindskog [12] theorem 1.27] If $X \sim RV(a_n, \mu)$ and $f : \mathbb{R}^d \to \mathbb{R}^m$ is continuous, homogeneous of degree γ (for some $\gamma > 0$) and respects origin exclusion, then*

$$nP(a_n^{-\gamma} f(X) \in \cdot) \xrightarrow{v} \mu \circ f^{-1}(\cdot \cap \mathbb{R}^m) \text{ on } \mathcal{B}(\overline{\mathbb{R}}^m - \{0\}).$$

This result is proved in [12] in the special case where f is linear and surjective but the same argument works in the general case. Indeed, a sufficient condition for f to respect origin exclusion is that it be continuous, homogeneous and surjective.

Theorem 5.6 *[Basrak, Davis and Mikosch [3], proposition A1] If $X \sim RV(a_n, \mu)$ and A is an $m \times d$ random matrix which is independent of X such that $\mathbb{E}(||A||^\beta) < \infty$, for some $\beta > \alpha$ (where α is the degree of homogeneity of μ), then $AX \sim RV(a_n, \mathbb{E}(\mu \circ A^{-1}(\cdot)))$.*

5.2 Estimates for Stochastic Integrals

Throughout this section $E = \{x \in \mathbb{R}^d, 0 < ||x|| < 1\}$ and $E^c = \{x \in \mathbb{R}^d, ||x|| \geqslant 1\}$, where $|| \cdot ||$ is the usual Euclidean norm. $\mathcal{P}$ denotes the predictable σ-algebra. Let $B = (B(t), t \geqslant 0)$ be a standard $\mathbb{R}^m$-valued Brownian motion and $N = (N(t, \cdot), t \geqslant 0)$ be a Poisson random measure on $\mathbb{R}^+ \times (\mathbb{R}^d - \{0\})$ which is independent of B. We assume that the intensity measure associated to N is of the form Leb$\otimes\nu$ where Leb denotes Lebesgue measure on $\mathbb{R}^+$ and ν is a Lévy measure on $\mathbb{R}^d - \{0\}$, i.e. $\int_{\mathbb{R}-\{0\}}(||x||^2 \wedge 1)\nu(dx) < \infty$. We will further assume that ν has support on the whole of $\mathbb{R}^d - \{0\}$. We denote by $\tilde{N}$ the associated compensated measure.
Now let (F, G, H, K) be a quadruple wherein

- F is an $\mathbb{R}^d$-valued predictable process.
- G is a $d \times m$ matrix-valued predictable process.
- H is a random mapping from $\mathbb{R}^+ \times E \to \mathbb{R}^d$, wherein each component is $\mathcal{P} \otimes \mathcal{B}(E)$-measurable.

- K is a random mapping from $\mathbb{R}^+ \times E^c \to \mathbb{R}^d$, wherein each component is $\mathcal{P} \otimes \mathcal{B}(E^c)$-measurable.

It is sometimes convenient to consider G as a $\mathbb{R}^{dm}$-valued vector with components $G^{ij}, i \leqslant i \leqslant d, 1 \leqslant j \leqslant m$ and associated norm $||G(t)|| = \sum_{i=1}^{d} \sum_{j=1}^{m} ||G^{ij}(t)||^2$, for each $t \geqslant 0$.
We impose the assumption that for all $t \geqslant 0$,

$$\int_0^t \left(||F(s)|| + ||G(s)||^2 + \int_{\hat{B}} ||H(s,x)||^2 \nu(dx) \right) ds < \infty \quad \text{a.s..}$$

We may then define the *Lévy-type stochastic integral* $M = (M(t), t \geqslant 0)$, where for each $t \geqslant 0, 1 \leqslant i \leqslant d$,

$$\begin{aligned} M^i(t) &:= \int_0^t F^i(s)ds + \sum_{j=1}^{m} \int_0^t G^{ij}(s)dB_j(s) + \int_0^t \int_E H^i(s,x)\tilde{N}(ds,dx) \\ &+ \int_0^t \int_{E^c} K^i(s,x)N(ds,dx) \\ &:= Y^i(t) + I_K^i(t), \end{aligned} \tag{5.9}$$

where $I_K^i(t) = \int_0^t \int_{\hat{B}^c} K^i(s,x)N(ds,dx)$ (see [2]).
Later on, we will want to apply theorem 5.4 to the process N and so we will need sufficient conditions for this to have finite moments for all time. This is given by the following:-

Condition A

- $h(t) := \sup_{0<|x|<1} \dfrac{||H(s,x)||}{||x||} < \infty$ for almost all $(t,\omega) \in \mathbb{R}^+ \times \Omega$.
- For each $t \geqslant 0, p > 2$,

$$\max\left\{ ||F(\cdot)||^p, ||G(\cdot)||^p, \int_{||x||<1} ||H(\cdot,x)||^p \nu(dx), h(\cdot)^p \right\} \in L^1(\Omega \times [0,t], P \otimes \text{Leb}).$$

Proposition 5.1 *If condition A holds, then* $\mathbb{E}(||Y(t)||^p) < \infty$ *for all* $t \geqslant 0$ *and all* $p \in \mathbb{N}$.

Proof By proposition 2.11 in [11], for each $p \geqslant 2$ there exists $C_p > 0$ such that

$$\begin{aligned}
\mathbb{E}\left(\sup_{0\leqslant s\leqslant t}||Y(s)||^p\right) &\leqslant C_p\left\{\mathbb{E}\left[\left(\int_0^t ||F(r)||dr\right)^p\right] + \mathbb{E}\left[\left(\int_0^t ||G(r)||^2 dr\right)^{\frac{p}{2}}\right]\right. \\
&+ \mathbb{E}\left[\left(\int_0^t \int_{||x||<1} ||H(r,x)||^2 \nu(dx)dr\right)^{\frac{p}{2}}\right] \\
&+ \left.\mathbb{E}\left[\left(\int_0^t \int_{||x||<1} ||H(r,x)||^p \nu(dx)dr\right)\right]\right\}.
\end{aligned}$$

Now a routine application of Hölder's inequality within each of the first three terms on the right hand side yields:

$$\begin{aligned}
\mathbb{E}\left(\sup_{0\leqslant s\leqslant t}||Y(s)||^p\right) &\leqslant D_p(t)\mathbb{E}\left[\int_0^t \left(||F(r)||^p + ||G(r)||^p + h(r)^p\right.\right. \\
&+ \left.\left.\int_{||x||<1} ||H(r,x)||^p \nu(dx)\right) dr\right],
\end{aligned}$$

where

$$D_p(t) = C_p \max\left\{1 + t^{p-1} + t^{\frac{p-2}{2}}\left(1 + \int_{||x||<1} ||x||^2 \nu(dx)\right)\right\},$$

and the result follows. □

5.3 The Main Result

We can now give our main result:

Theorem 5.7 *Let $M = (M(t), t \geqslant 0)$ be a Lévy type stochastic integral of the form (5.9) and assume that condition A holds. We also assume the following:*

1. *$\nu \sim RV(a_n, \mu)$.*
2. *$K(t,x) = A(t)g(x)$, for each $t \geqslant 0, x \in E^c$, where*
   - *A is a random $d \times l$ matrix with predictable entries and $\mathbb{E}(||A(t)||^\beta < \infty$, for all $t \geqslant 0$, for some $\beta > \alpha$, where α is the degree of homogeneity of $\mu \circ g^{-1}$.*
   - *$g : \mathbb{R}^d \to \mathbb{R}^l$ is continuous, homogeneous of degree γ (for some $\gamma > 0$) and respects origin exclusion.*

3. The process $(A(t), t \geqslant 0)$ is independent of $(Y(t), t \geqslant 0)$ and of the Poisson random measure N

Then for each $t > 0$,

$$nP(a_n^{-\gamma} M(t) \in \cdot) \xrightarrow{v} \mathbb{E}[\eta(t)(\cdot \cap \mathbb{R}^l)] \text{ on } \mathcal{B}(\overline{\mathbb{R}}^l - \{0\}),$$

where $\eta(t) = t(\mu \circ K(t, \cdot)^{-1})$.

Proof. For each $t \geqslant 0$ define $J(t) = \int_{||x||>1} g(x)N(t, dx)$, so that $J = (J(t), t \geqslant 0)$ is a compound Poisson process with Lévy measure $\nu \circ g^{-1}$. We can write each $J(t) = \sum_{j=1}^{N(t)} g(Y_j)$ where $(Y_n, n \in \mathbb{N})$ is a sequence of i.i.d random variables with common law $\dfrac{\nu(\cdot \cap E^c)}{\nu(E^c)}$ and $(N(t), t \geqslant 0)$ is an independent Poisson process with intensity $\nu(E^c)$. It follows from theorem 5.5 that for each $m \in \mathbb{N}$,

$$nP(a_n^{-\gamma} g(Y_m) \in \cdot) \xrightarrow{v} \frac{1}{\nu(E^c)}[\mu \circ g^{-1}(\cdot \cap \mathbb{R}^l)] \text{ on } \mathcal{B}(\overline{\mathbb{R}}^l - \{0\}).$$

Now by assumption (1) and theorem 5.3, for each $t \geqslant 0$ we have

$$nP(a_n^{-\gamma} J(t) \in \cdot) \xrightarrow{v} t\mu \circ g^{-1}(\cdot \cap \mathbb{R}^l) \text{ on } \mathcal{B}(\overline{\mathbb{R}}^l - \{0\}).$$

The result now follows from theorems 5.2, 5.4 and 5.6 via proposition 5.1. □

6 Appendix

The proof of the following result is due to Gennady Samorodnitsky.

Theorem 6.1 *If X and Y are independent real-valued random variables with $\overline{F_X} \in \mathcal{R}_{-\alpha}$ and $\overline{F_Y} \in \mathcal{R}_{-\beta}$, where $\alpha, \beta > 0$, then $\overline{F_{X+Y}} \in \mathcal{R}_{-\rho}$, where $\rho = \min\{\alpha, \beta\}$.*

Proof. It is sufficient to prove the result when X and Y are both non-negative. We aim to prove that

$$\overline{F_{X+Y}}(\lambda) \sim \overline{F_X}(\lambda) + \overline{F_Y}(\lambda),$$

as $\lambda \to \infty$. The result then follows easily, by the definition of regular variation.

First observe, that

$$\begin{aligned}
P(X+Y>\lambda) &\geqslant P(\{X>\lambda\}\cup\{Y>\lambda\}) \\
&= P(X>\lambda)+P(Y>\lambda)-P(X>\lambda)P(Y>\lambda) \\
&\sim P(X>\lambda)+P(Y>\lambda),
\end{aligned}$$

as $\lambda \to \infty$. Hence

$$\liminf_{\lambda\to\infty} \frac{P(X+Y>\lambda)}{P(X>\lambda)+P(Y>\lambda)} \geqslant 1.$$

To obtain the reverse inequality, let $0<\epsilon<1$, then

$$\begin{aligned}
P(X+Y>\lambda) &\leqslant P(\{X>(1-\epsilon)\lambda\}\cup\{Y>(1-\epsilon)\lambda\}\cup(\{X\geqslant\epsilon\lambda\}\cap\{Y\geqslant\epsilon\lambda\})) \\
&\leqslant P(X>(1-\epsilon)\lambda)+P(Y>(1-\epsilon)\lambda)+P(X\geqslant\epsilon\lambda, Y\geqslant\epsilon\lambda) \\
&\sim P(X>(1-\epsilon)\lambda)+P(Y>(1-\epsilon)\lambda),
\end{aligned}$$

as $\lambda\to\infty$.
From this we see that

$$\begin{aligned}
&\limsup_{\lambda\to\infty} \frac{P(X+Y>\lambda)}{P(X>\lambda)+P(Y>\lambda)} \\
\leqslant\ &\limsup_{\lambda\to\infty} \frac{P(X+Y>\lambda)}{P(X>(1-\epsilon)\lambda)+P(Y>(1-\epsilon)\lambda)} \\
\times\ &\limsup_{\lambda\to\infty} \frac{P(X>(1-\epsilon)\lambda)+P(Y>(1-\epsilon)\lambda)}{P(X>\lambda)+P(Y>\lambda)} \\
\leqslant\ &(1-\epsilon)^{-\max\{\alpha,\beta\}}.
\end{aligned}$$

The required result follows on taking the limit as $\epsilon \downarrow 0$. □

Related results to theorem 6.1 can be found in [9], p.278, [8], lemma 1.3.1 and [15], proposition 4.1.

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